CATENARY CURVE

Rod Deakin
DUNSBOROUGH, WA, 6281, Australia
email: randm.deakin@gmail.com
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Abstract

The *catenary* is the curve in which a uniform chain or cable hangs freely under the force of gravity from two supports. It is a U-shaped curve symmetric about a vertical axis through its low-point and was first described mathematically by Leibniz, Huygens and Johann Bernoulli in 1691 responding to a challenge put out by Jacob Bernoulli (Johann's elder brother) to find the equation of the 'chain curve'. Every person viewing power lines hanging between supporting poles is seeing a catenary, a curve whose name is derived from the Latin word *catena,* meaning chain, and the catenary's mathematical 'discovery' is due to Galileo's claim – proved incorrect by Bernoulli and others – that a hanging chain was parabolic. This paper gives a mathematical derivation of the catenary with examples.

Keywords

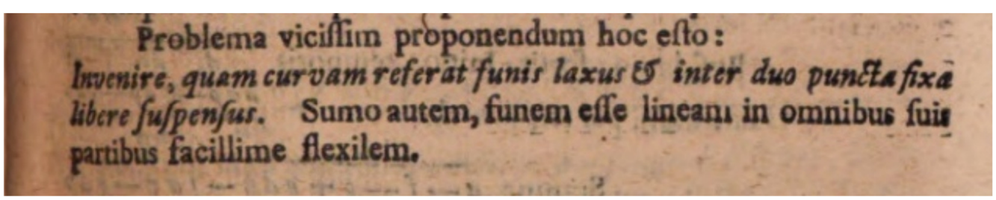
Catenary, hanging chain, hanging cable

Introduction

Galileo Galelai in his *Discorsi*[1] (1638) describes two methods of drawing a parabola, one simulates a ballistic trajectory with a smooth ball on an inclined mirror and the other is explained as:

> "Drive two nails into a wall at a convenient height and at the same level; ... Over these two nails hang a light chain ... This chain will assume the form of a parabola, ..."

Unfortunately, whilst it is close, the hanging chain does not assume the form of a parabola and Galileo's assertion became a discussion point for followers of his work. Christiaan Huygens (1629–1695) in correspondence in 1646 with the French monk Marin Mersenne (1588–1648) of Mersenne primes fame gave a proof that a hanging chain was not a parabola (Huygens 1646, Bukowski 2008) and it was Huygens in a letter of November 1690 to the German polymath Gottfried Leibniz (1646-1716) who first used the Latin term *catenaria* to describe the curve (Bukowski 2008). Other prominent mathematicians of the time also studied the hanging chain problem, but it was the Swiss mathematician Jacob Bernoulli (1654–1705) who brought about the discovery of its mathematical description. In *Acta Eruditorum*[2] of May 1690 (pp. 217-219) Jacob Bernoulli gave a solution to the isochrone problem of constructing the curve along which a body will fall in the same amount of time from any starting position (Barnett 2004, Peiffer 2006), and after this solution he wrote:


Problema viciſſim proponendum hoc eſto:
Invenire, quam curvam referat funis laxus & inter duo puncta fixa libere ſuſpenſus. Sumo autem, funem eſſe lineam in omnibus ſuis partibus facillime flexilem.

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[1] *Discorsi e dimostrazioni mathematiche, intorno à due nuoue scienze* (translated from Italian as Dialogues concerning two new sciences) published in 1638 was Galileo's final book and a scientific testament covering much of his work in physics over the preceding thirty years. Appendix C has extracts from a translation.

[2] *Acta Eruditorum* (Latin for "Acts of the Erudite") was the first scientific journal of the German-speaking lands of Europe, published from 1682 to 1782. Appendix C has extracts.

That can be translated as (Mukhopadhyay 2001)

> And now let this problem be proposed: To find the curve assumed by a loose string hung freely from two fixed points. And let this string be flexible and of uniform cross-section.

In July 1690 in *Acta Eruditorum,* Leibniz takes up Jacob Bernoulli's challenge and proposes that solutions be produced in one years' time and he receives solutions from Huygens and Jacob's younger brother Johann Bernoulli. And together with his own, three solutions to the hanging chain problem are published in *Acta Eruditorum* in June and September 1691 (Leibniz 1691). Interestingly, Jacob Bernoulli did not provide a solution to the hanging chain problem – a point mentioned often by his younger brother Johann in a sometimes-bitter rivalry that lasted beyond Jacob's death in 1705 – but in the year following the first solution Jacob and others solved several variations of this problem (Barnett 2004). Both Leibniz and Johan Bernoulli use the new methods of calculus developed by Leibniz in their solutions but it is Johann Bernoulli's explanations of the forces acting on the chain that enable him to show that the curve satisfies the differential equation $dy/dx = s/a$ where s represents the arc length from the vertex to a point P on the curve and a is a constant depending on the weight per unit length of the chain (Barnett 2004, Bernoulli 1692). We still use Bernoulli's reasoning today in modern developments of the equations associated with the catenary although there is no mention of hyperbolic functions in 1691 nor is there reference to the exponential function $y = e^x$ where $e = 2.7182818284\ldots$ is a mathematical constant and the base of the natural logarithms.

In this paper two methods of derivation of the equations for the catenary are shown and these are followed by an example and then a solution to *Curly's Conundrum No.15* involving a parabolic arc and a comparison with a catenary.

Nomenclature

The following notation has been used, noting that forces are vector quantities having both magnitude and direction and are denoted by **bold** uppercase letters. The magnitude of a vector quantity is shown in *italics,* e.g., the tension force is denoted by **T** and its magnitude by T.

Symbol	**Meaning**	**Definition**
a	parameter in the catenary equation	
d	depth of catenary	
e	base of natural logarithms	$e = 2.7182818284\ldots$
g	acceleration due to gravity $\left(\text{m/s}^2\right)$	
h	horizontal distance between catenary supports	
v	Vertical distance between catenary supports	
kg	kilogram (unit of mass)	
L	length of catenary between supports (m)	
M, m, m	mass of chain (kg), mass of segment of chain (kg), metre (unit of length)	$m = \lambda s$
s, s	arc length (m), second (unit of time)	
T, T	Tension (a pulling force), magnitude of tension	
W, W	Weight (a force), magnitude of weight	
x,y	rectangular coordinates of P	
κ	curvature	$\kappa = d\psi/ds$
λ	mass per unit length of chain $\left(\text{kg/m}\right)$	$\lambda = M/L$
ρ	radius of curvature	$\rho = 1/\kappa = ds/d\psi$
ψ	angle between tangent and x-axis (radians)	

Equations for the Catenary

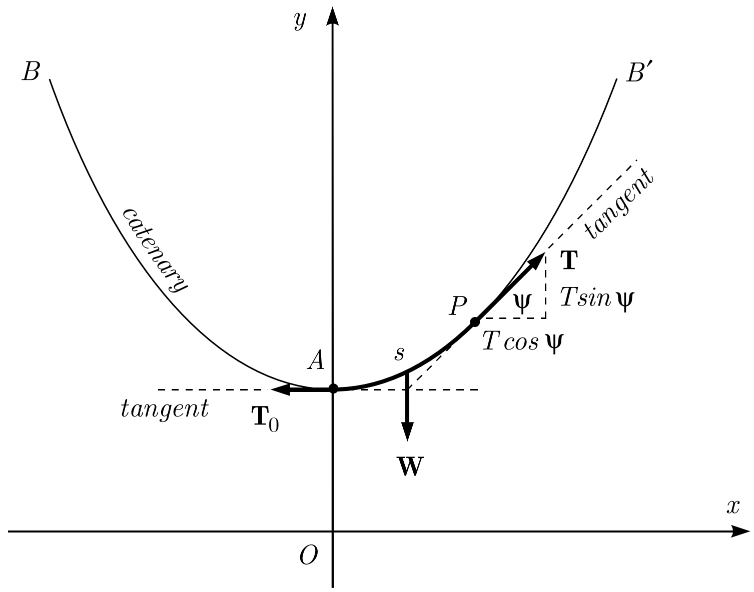


Figure 1. A catenary formed by a chain of length L supported at B and B'.

In Figure 1, B and B' are the supports of a hanging chain or catenary. The low point is at A and P is a point on the catenary at a distance s from A. The chain of length L is of uniform cross-section and density and has a mass M, hence $\lambda = M/L$ is a constant denoting mass per unit length of the chain.

The chain is hanging in equilibrium; hence the segment AP of length s is also in equilibrium and the three forces acting on this segment – $\mathbf{T}_0$ (tension at A acting along the tangent at A that is parallel to the *x-axis*), $\mathbf{T}$ (tension at P acting upwards along the tangent at P) and the weight $\mathbf{W}$ (acting vertically downwards parallel to the y-axis) – must sum to zero. Equating the magnitudes of the force components gives the following, noting that the magnitude of $\mathbf{W}$ is $W = mg = \lambda sg$

$$\begin{aligned} T\sin\psi &= \lambda sg \\ T\cos\psi &= T_0 \end{aligned} \tag{1}$$

And division of the members of (1) gives, noting that the gradient of the catenary is $dy/dx = \tan\psi$

$$\frac{dy}{dx} = \tan\psi = \frac{\lambda sg}{T_0} \tag{2}$$

Also, squaring and adding the members of (1) gives $T^2\left(\sin^2\psi + \cos^2\psi\right) = T_0^2 + \left(\lambda sg\right)^2$ and since $\sin^2 A + \cos^2 A = 1$ then

$$T^2 = T_0^2 + \left(\lambda sg\right)^2 \tag{3}$$

Now, define a parameter a having units of length

$$a = \frac{T_0}{\lambda g} \tag{4}$$

Noting that a dimensional analysis with M = mass (kg), L = length (m) and T = time (s) gives

$$T_0 = \mathrm{MLT^{-2}},\ \lambda = \mathrm{ML^{-1}},\ g = \mathrm{LT^{-2}}\ \text{and}\ a = \left(\mathrm{MLT^{-2}}\right)\left(\mathrm{M^{-1}L}\right)\left(\mathrm{L^{-1}T^{2}}\right) = \mathrm{L}$$

Then substituting (4) into (2) and (3) gives

$$\frac{dy}{dx} = \tan\psi = \frac{s}{a} \tag{5}$$

$$T = \lambda g \sqrt{a^2 + s^2} \tag{6}$$

Solving the differential equation (5) to obtain $y = y(x)$ and then deriving other equations of the catenary can be done in two methods set out below.

Method A (the usual method found in documents on the catenary)

Following Nahin (2004) and noting that the differential arc length $ds = \sqrt{dx^2 + dy^2}$ (Pythagoras) and $\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$, differentiate (5) with respect to *x,* i.e., $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{s}{a}\right) = \frac{d}{ds}\left(\frac{s}{a}\right)\frac{ds}{dx}$ that simplifies to the 2nd order differential equation

$$\frac{d^2y}{dx^2} = \frac{1}{a}\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \tag{7}$$

To solve this differential equation define a variable p as

$$p = \frac{dy}{dx}\ \text{and so}\ \frac{dp}{dx} = \frac{d^2y}{dx^2}$$

and (7) becomes

$$\frac{dp}{dx} = \frac{1}{a}\sqrt{1 + p^2}\quad \text{or}\quad dx = \frac{a\,dp}{\sqrt{1 + p^2}}$$

Which can now be easily integrated and from a table of standard integrals

$$x = a\int \frac{dp}{\sqrt{1 + p^2}} = a\ln\left(p + \sqrt{1 + p^2}\right) + C_1 = a\sinh^{-1} p + C_1$$

where ln denotes the natural logarithm and $\ln x \equiv \log_e x$ and $\sinh^{-1}$ is the inverse hyperbolic sine function defined in Appendix A. C_1 is a constant of integration that that can be evaluated by considering Figure 1 and noting that when $x = 0$, $p = \frac{dy}{dx} = \tan\psi = 0$ and $\ln(1) = 0$ then $C_1 = 0$. Note also that $\sinh x$ is the hyperbolic sine of x and $\sinh\left(\sinh^{-1} x\right) = x$ (see Appendix A) hence

$$p = \sinh\left(\frac{x}{a}\right) = \frac{dy}{dx} \tag{8}$$

y can now be obtained by a second integration and standard integral results can be used to give

$$y = \int \sinh\left(\frac{x}{a}\right)dx = a\cosh\left(\frac{x}{a}\right) + C_2$$

Where cosh is the hyperbolic cosine function and C_2 is another constant of integration that can be made to equal zero if $y = a$ when $x = 0$. This means that if the low point of the catenary is at $(0, a)$ then the catenary equation $y = y(x)$ is

$$y = a\cosh\left(\frac{x}{a}\right) \tag{9}$$

The equations for arc length $s = s(x)$, equations $y = y(\psi)$ and $x = x(\psi)$, the inclination of the tangent $\psi = \psi(x)$ and the tension $T = T(y)$ are set out below.

Equating (5) and (8) gives $s = s(x)$ as

$$s = a\sinh\left(\frac{x}{a}\right) \tag{10}$$

The equations $y = y(\psi)$ and $x = x(\psi)$ can be obtained as follows.

From (5), $s = a\tan\psi$ and $s^2 = a^2\tan^2\psi = a^2\sec^2\psi - a^2$ since $\tan^2 A = \sec^2 A - 1$. Hence $a^2 + s^2 = a^2\sec^2\psi$. Also, squaring (9) and (10) and subtracting one from the other gives $y^2 - s^2 = a^2\left(\cosh^2\left(\frac{x}{a}\right) - \sinh^2\left(\frac{x}{a}\right)\right)$ and since $\cosh^2 A - \sinh^2 A = 1$ (see Appendix A) then $a^2 + s^2 = y^2$. Equating these two results gives $y^2 = a^2\sec^2\psi$ and $y = y(\psi)$ as

$$y = a\sec\psi \tag{11}$$

From (10) $\frac{s}{a} = \sinh\left(\frac{x}{a}\right)$ and from (5) $\frac{s}{a} = \tan\psi$ hence $\tan\psi = \sinh\left(\frac{x}{a}\right)$ or $x = a\sinh^{-1}(\tan\psi)$ and using the identity (57) in Appendix A gives $x = x(\psi)$ as

$$x = a\ln\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\psi\right) = a\ln\left(\tan\psi + \sec\psi\right) \tag{12}$$

An equation for $\psi = \psi(x)$ can be obtained from the hyperbolic half-angle formula $\tanh\left(\frac{A}{2}\right) = \frac{\cosh A - 1}{\sinh A}$ and $\tanh\left(\frac{x}{2a}\right) = \frac{\cosh\left(\frac{x}{a}\right) - 1}{\sinh\left(\frac{x}{a}\right)}$ and using (9) and (10) gives $\tanh\left(\frac{x}{2a}\right) = \frac{\left(\frac{y}{a}\right) - 1}{\left(\frac{s}{a}\right)} = \left(\frac{y}{a}\right)\left(\frac{a}{s}\right) - \frac{a}{s}$ and using (5) and (11) gives $\tanh\left(\frac{x}{2a}\right) = \frac{\sec\psi}{\tan\psi} - \frac{1}{\tan\psi}$ and the right-hand side reduces to $\frac{1 - \cos\psi}{\sin\psi}$. Now using the trigonometric half-angle formula $\tan\left(\frac{A}{2}\right) = \frac{1 - \cos A}{\sin A}$ gives $\tanh\left(\frac{x}{2a}\right) = \tan\left(\frac{\psi}{2}\right)$ and so

$$\psi = 2\tan^{-1}\left[\tanh\left(\frac{x}{2a}\right)\right] \tag{13}$$

Finally, the tension $T = T(y)$ can be obtained by substituting $y^2 = a^2 + s^2$ (see derivation above) into (6) to give

$$T = \lambda g\, y \tag{14}$$

Note that a dimensional analysis gives $T = \left(\mathrm{ML}^{-1}\right)\left(\mathrm{LT}^{-2}\right)\left(\mathrm{L}\right) = \mathrm{MLT}^{-2}$ giving the units of T as $\left(\mathrm{kg\ m}\right)/\mathrm{s}^2$ as expected.

Method B (an alternative method not often found in documents on the catenary)

Following Lamb (1942), write (5) as

$$s = a\tan\psi \tag{15}$$

and differentiate this equation with respect to ψ to give

$$\frac{ds}{d\psi} = a\sec^2\psi \tag{16}$$

Now, an element of arc length $ds = \sqrt{dx^2 + dy^2}$ (Pythagoras) where $dx = ds\cos\psi$ and $dy = ds\sin\psi$ and

$$\frac{dx}{ds} = \cos\psi,\quad \frac{dy}{ds} = \sin\psi,\quad \frac{dy}{dx} = \tan\psi, \tag{17}$$

and, using the chain rule we have $\dfrac{dx}{d\psi} = \dfrac{dx}{ds}\dfrac{ds}{d\psi}$ and with (16) and (17)

$$\frac{dx}{d\psi} = \cos\psi\left(a\sec^2\psi\right) = a\sec\psi \tag{18}$$

Similarly, $\dfrac{dy}{d\psi} = \dfrac{dy}{ds}\dfrac{ds}{d\psi}$ giving

$$\frac{dy}{d\psi} = \sin\psi\left(a\sec^2\psi\right) = a\sec\psi\tan\psi \tag{19}$$

And x and y can now be obtained from integration of (18) and (19) with standard integral results giving

$$x = a\ln\left(\sec\psi + \tan\psi\right) + C_1 \tag{20}$$

$$y = a\sec\psi + C_2 \tag{21}$$

where C_1 and C_2 are constants of integration. These constants will both be equal to zero when the low point of the catenary is at $\left(0,a\right)$ (see Figure 1) and the equations $y = y\left(\psi\right)$ and $x = x\left(\psi\right)$ are

$$x = a\ln\left(\sec\psi + \tan\psi\right) \tag{22}$$

$$y = a\sec\psi \tag{23}$$

Now rearranging (22) and raising both sides as powers of the base e, noting $e^{\ln x} = x$ gives

$$e^{x/a} = \sec\psi + \tan\psi \tag{24}$$

Taking the reciprocal of (24) and using the laws of exponents gives $e^{-x/a} = \dfrac{1}{\sec\psi + \tan\psi}$ and noting that the trigonometric identity $\sec^2\psi - \tan^2\psi = 1$ can be written as $\left(\sec\psi + \tan\psi\right)\left(\sec\psi - \tan\psi\right) = 1$ then

$$e^{-x/a} = \sec\psi - \tan\psi \tag{25}$$

Adding and subtracting (24) and (25) and using the definitions in Appendix A gives

$$\sec\psi = \frac{e^{x/a} + e^{-x/a}}{2} = \cosh\left(\frac{x}{a}\right) \tag{26}$$

$$\tan\psi = \frac{e^{x/a} - e^{-x/a}}{2} = \sinh\left(\frac{x}{a}\right) \tag{27}$$

And using (26) in (23) and (27) in (15) gives equation for $y = y(x)$ and $s = s(x)$ as

$$y = a\cosh\left(\frac{x}{a}\right) \tag{28}$$

$$s = a\sinh\left(\frac{x}{a}\right) \tag{29}$$

An equation for $\psi = \psi(x)$ is obtained from (27) as

$$\psi = \tan^{-1}\left[\sinh\left(\frac{x}{a}\right)\right] \tag{30}$$

Finally, the tension $T = T(y)$ can be obtained as follows. First, squaring (15) and (23) and subtracting one from the other gives $y^2 - s^2 = a^2\left(\sec^2\psi - \tan^2\psi\right) = a^2$, or $y^2 = a^2 + s^2$ and using this result in (6) gives

$$T = \lambda g\, y \tag{31}$$

Catenaries for different values of a

The parameter a determines the shape of the catenary and as a increases the catenary becomes shallower.

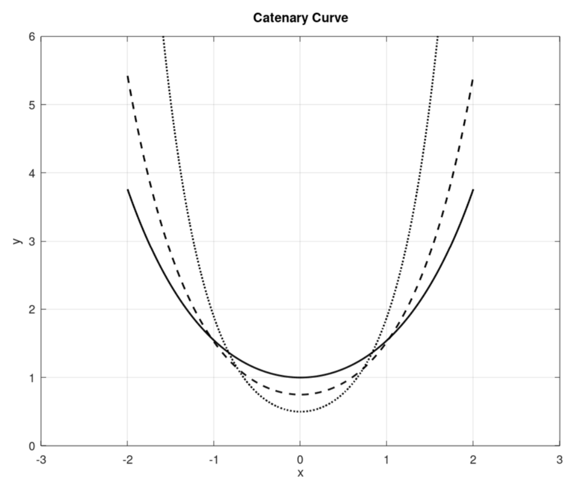


Figure 2. The catenary $y = a\cosh\left(\frac{x}{a}\right)$ for $a = 1,\ 0.75,\ 0.5$ (solid, dashed, dotted)

Geometry of the catenary

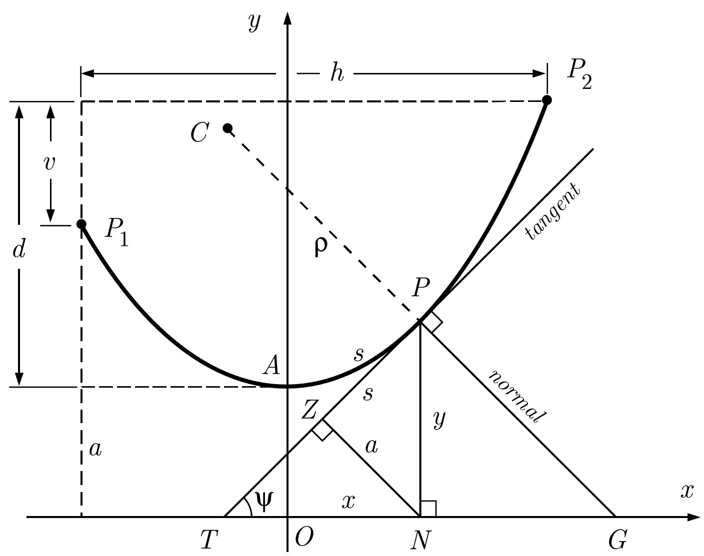


Figure 3. The catenary $y = a\cosh\left(\dfrac{x}{a}\right)$ of length L between supports P_1 and P_2

In Figure 3, P_1 and P_2 are supports of a hanging chain or catenary of length L. The low point is at A and P is a point on the catenary at a distance s from A. The y-axis passes through A and intersects the x-axis at O. The distance $a = OA$ is a constant in the catenary equation $y = a\cosh\left(\dfrac{x}{a}\right)$.

P_1 with coordinates $\left(x_1, y_1\right)$ is to the left of the y-axis and $P_2\left(x_2, y_2\right)$ is to the right and they are separated by horizontal and vertical distances h and v respectively, and the depth d of the catenary is the vertical distance from the low point A to the highest support.

The tangent to the catenary at $P\left(x, y\right)$ intersects the x-axis at T at an angle ψ and the normal to the catenary at P intersects the x-axis at G. The centre of curvature C lies on the normal and $\rho = CP$ is the radius of curvature. By definition, the curvature κ (kappa) of a curve $y = f\left(x\right)$ at any point P on the curve, is the rate of change of direction of the tangent to the curve with respect to the arc length and $\kappa = \dfrac{d\psi}{ds}$. The radius of curvature $\rho = \dfrac{1}{\kappa} = \dfrac{ds}{d\psi}$ and from (16), (23) and (26) we may write (Yates 1959)

$$\rho = a\sec^2\psi = y\sec\psi = \frac{y^2}{a} = a\cosh^2\left(\frac{x}{a}\right) \tag{32}$$

For P with $x = ON$ and $y = NP$ we have from (23), (15) and (32)

$$NZ = y\cos\psi = a, \quad PZ = a\tan\psi = s, \quad PG = y\sec\psi = \rho \tag{33}$$

And, as previously shown in derivations of (11) and (31), and also from the right-angled triangle PZN,

$$y^2 = a^2 + s^2 \tag{34}$$

Determining the parameter a

Following the *Wikipedia* entry Catenary (Catenary 2019), suppose the horizontal distance h and vertical distance v between the catenary supports P_1, P_2 are known and the length L of the catenary is known. How can the parameter a be determined?

Given that the coordinates of P_1, P_2 are $(x_1, y_1), (x_2, y_2)$ respectively and that P_1 is to the left of P_2 and the y-axis passes through A (the low point of the catenary) and intersects the x-axis at O (see Figure 3); the distance $a = OA$ is the unknown parameter. Distances s along the catenary to the left of the y-axis will be negative and those to the right will be positive. Also assume $y_2 > y_1$ (i.e., P_2 is higher than P_1). Hence h, v and L are given by

$$h = x_2 - x_1, \quad v = y_2 - y_1, \quad L = s_2 - s_1 \tag{35}$$

where s_1, s_2 are distances A to P_1, P_2 respectively.

Now using equations $y = y(x)$ and $s = s(x)$ (see equations (9), (10)) write

$$\begin{aligned} v &= a\cosh\left(\frac{x_2}{a}\right) - a\cosh\left(\frac{x_1}{a}\right) \\ L &= a\sinh\left(\frac{x_2}{a}\right) - a\sinh\left(\frac{x_1}{a}\right) \end{aligned} \tag{36}$$

Squaring both equations gives

$$\begin{aligned} v^2 &= a^2\left(\cosh^2\left(\frac{x_1}{a}\right) + \cosh^2\left(\frac{x_2}{a}\right) - 2\cosh\left(\frac{x_1}{a}\right)\cosh\left(\frac{x_2}{a}\right)\right) \\ L^2 &= a^2\left(\sinh^2\left(\frac{x_1}{a}\right) - \sinh^2\left(\frac{x_2}{a}\right) - 2\sinh\left(\frac{x_1}{a}\right)\sinh\left(\frac{x_2}{a}\right)\right) \end{aligned}$$

and subtracting one from the other noting that $\cosh^2 A - \sinh^2 A = 1$ gives

$$L^2 - v^2 = a^2\left(2\cosh\left(\frac{x_1}{a}\right)\cosh\left(\frac{x_2}{a}\right) - 2\sinh\left(\frac{x_1}{a}\right)\sinh\left(\frac{x_2}{a}\right) - 2\right)$$

Now using the hyperbolic trigonometric identities: $2\sinh A \sinh B = \cosh(A + B) - \cosh(A - B)$ and $2\cosh A \cosh B = \cosh(A + B) + \cosh(A - B)$ gives

$$L^2 - v^2 = 2a^2\left(\cosh\left(\frac{x_1 - x_2}{a}\right) - 1\right)$$

but $x_1 - x_2 = -h$ and $\cosh(-h) = \cosh h$, and $\cosh A - 1 = 2\sinh^2\left(\frac{A}{2}\right)$ so we can write

$$L^2 - v^2 = 4a^2\sinh^2\left(\frac{h}{2a}\right) \quad \text{or} \quad \sqrt{L^2 - v^2} = 2a\sinh\left(\frac{h}{2a}\right) \tag{37}$$

This is a transcendental[3] equation in a and cannot be solved algebraically. Instead it must be solved by numerical methods and one such method is Newton-Raphson iteration, where, for $f(a) = 0$ a value of a may be found from the iterative equation

[3] A transcendental equation contains one or more transcendental functions and such functions cannot be expressed in terms of polynomials or solved by algebraic methods. Hyperbolic functions are transcendental functions.

$$a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)} \tag{38}$$

where n denotes the n^{th} iteration and $f(a)$ from (37) is

$$f(a) = 2a\sinh\left(\frac{h}{2a}\right) - \sqrt{L^2 - v^2} \tag{39}$$

and the derivative $f'(a) = \frac{d}{da}\{f(a)\}$ is given by

$$f'(a) = 2\left\{\sinh\left(\frac{h}{2a}\right) - \frac{h}{2a}\cosh\left(\frac{h}{2a}\right)\right\} \tag{40}$$

An initial value for a (for $n = 1$) is chosen and the functions $f(a_1)$ and $f'(a_1)$ evaluated from (39) and (40) using a_1. The next value, a_2 (a for $n = 2$) is now computed from (38) and this process repeated to obtain values $a_3, a_4, \ldots$ This iterative process can be concluded when the difference between a_{n+1} and a_n reaches an acceptably small value.

After solving for the parameter a, the x,y coordinates of the supports can be determined if x_2 is known. Following Cella (1999), x_2 can be determined as follows.

Using (36) and (35) we may write, noting that $\cosh(-x) = \cosh x$ and $\sinh(-x) = -\sinh x$

$$\begin{aligned} v &= a\cosh\left(\frac{x_2}{a}\right) - a\cosh\left(\frac{x_2 - h}{a}\right) = a\cosh\left(\frac{x_2}{a}\right) - a\cosh\left(\frac{h - x_2}{a}\right) \\ L &= a\sinh\left(\frac{x_2}{a}\right) - a\sinh\left(\frac{x_2 - h}{a}\right) = a\sinh\left(\frac{x_2}{a}\right) + a\sinh\left(\frac{h - x_2}{a}\right) \end{aligned}$$

and dividing one equation by the other gives

$$\frac{v}{L} = \frac{\cosh\left(\frac{x_2}{a}\right) - \cosh\left(\frac{h - x_2}{a}\right)}{\sinh\left(\frac{x_2}{a}\right) + \sinh\left(\frac{h - x_2}{a}\right)}$$

Using the hyperbolic identities $\cosh A - \cosh B = 2\sinh\left(\frac{A+B}{2}\right)\sinh\left(\frac{A-B}{2}\right)$ and

$\sinh A + \sinh B = 2\sinh\left(\frac{A+B}{2}\right)\cosh\left(\frac{A-B}{2}\right)$ gives $\frac{v}{L} = \frac{2\sinh\left(\frac{h}{2a}\right)\sinh\left(\frac{2x_2 - h}{2a}\right)}{2\sinh\left(\frac{h}{2a}\right)\cosh\left(\frac{2x_2 - h}{2a}\right)}$ and

$$\frac{v}{L} = \tanh\left(\frac{2x_2 - h}{2a}\right)$$

Solving this equation for x_2/a gives

$$\frac{x_2}{a} = \frac{h}{2a} + \tanh^{-1}\left(\frac{v}{L}\right) \quad \text{or} \quad x_2 = \frac{h}{2} + a\tanh^{-1}\left(\frac{v}{L}\right) \tag{41}$$

Now, having obtained x_2, then $x_1 = x_2 - h$ and y_1, y_2 and s_1, s_2 follow from (9) and (10).

Example

Suppose that we wish to find the sag below the upper support P_2 of a cable 110 m long suspended between supports P_1, P_2 that are 100 m apart horizontally and 20 m vertically (Cella 1999).

In Figure 4, h and v are the horizontal and vertical distances between supports P_1 and P_2. L is the length of the catenary and $a = OA$

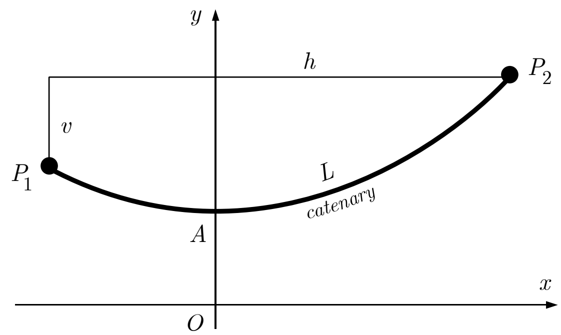


Figure 4

Solution

1 Solve for a using Newton-Raphson iteration (38), (39) and (40) with $h = 100$, $v = 20$ and $L = 110$

$$a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)} \quad \text{where} \quad \begin{aligned} f(a) &= 2a\sinh\left(\frac{h}{2a}\right) - \sqrt{L^2 - v^2} \\ f'(a) &= 2\left\{\sinh\left(\frac{h}{2a}\right) - \frac{h}{2a}\cosh\left(\frac{h}{2a}\right)\right\} \end{aligned}$$

n	a_n	$f(a)$	$f'(a)$	$c = f(a)/f'(a)$	$a_{n+1} = a_n - c$
1	50.000000000	9.353581100	-0.735758882	-12.712834768	62.712834768
2	62.712834768	2.769714918	-0.359840373	-7.697065504	70.409900272
3	70.409900272	0.452622368	-0.250993660	-1.803321916	72.213222188
4	72.213222188	0.017347334	-0.232085745	-0.074745367	72.287967555
5	72.287967555	0.000027743	-0.231343945	-0.000119921	72.288087476
6	72.288087476	0.000000000	-0.231342758	0.000000000	72.288087476

Table 1. Values for Newton-Raphson iteration

The solution $a = 72.288087476$ is achieved after 6 iterations using a starting value $a_1 = 50$

2 Solve for x_2 using (41)

$$x_2 = \frac{h}{2} + a\tanh^{-1}\left(\frac{v}{L}\right) = 63.291060536$$

3 Solve for x_1 using (35) then y_1, y_2 and s_1, s_2 using (9) and (10) respectively

$$x_1 = x_2 - h = -36.708939464$$

$$y_1 = a\cosh\left(\frac{x_1}{a}\right) = 81.81078082, \qquad y_2 = a\cosh\left(\frac{x_2}{a}\right) = 101.810780818$$

$$s_1 = a\sinh\left(\frac{x_1}{a}\right) = -38.30713076 \qquad s_2 = a\sinh\left(\frac{x_2}{a}\right) = 71.69286924$$

4 The sag of the cable below P_2 is the vertical distance $y_2 - a = 29.522693342$

It should be noted that in solving for a using Newton-Raphson iteration care must be taken in selecting an appropriate starting value a_1. An approximation for a can be obtained as follows

From (37) we may write

$$\frac{\sqrt{L^2 - v^2}}{2a} = \sinh\left(\frac{h}{2a}\right)$$

Using the series expansion for $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$ gives

$$\frac{\sqrt{L^2 - v^2}}{2a} = \frac{h}{2a} + \frac{1}{6}\frac{h^3}{\left(2a\right)^3} + \frac{1}{120}\frac{h^5}{\left(2a\right)^5} + \frac{1}{5040}\frac{h^7}{\left(2a\right)^7} + \cdots$$

Ignoring terms involving powers of 5 or greater and then rearranging gives

$$a \approx \frac{h}{\sqrt{24}}\sqrt{\frac{h}{\sqrt{L^2 - v^2} - h}} \tag{42}$$

In the example above where $h = 100$, $v = 20$ and $L = 110$ a starting value for a in the Newton-Raphson iteration could have been $a_1 = \frac{100}{\sqrt{24}}\sqrt{\frac{100}{\sqrt{110^2 - 20^2} - 100}} = 71.4$ rather than $a_1 = 50$. This would have meant fewer iterations before convergence on $a = 72.288087476$.

An alternative method of determining the parameter *a*

Cella (1999) published a short article on the practical determination of the parameters of a catenary and derived several useful equations that are set out below. Equation (41) above was derived by Cella (1999, eq. 7) as an interim step in the development a transcendental equation

Using (9), (10) and (35) we may write, noting that $\cosh\left(-x\right) = \cosh x$ and $\sinh\left(-x\right) = -\sinh x$

$$\begin{aligned} y_2 &= a\cosh\left(\frac{x_2}{a}\right), \quad & y_2 - v &= a\cosh\left(\frac{x_2 - h}{a}\right) \quad \text{or} \quad & y_2 - v &= a\cosh\left(\frac{h - x_2}{a}\right) \\ s_2 &= a\sinh\left(\frac{x_2}{a}\right), \quad & s_2 - L &= a\sinh\left(\frac{x_2 - h}{a}\right) \quad \text{or} \quad & L - s_2 &= a\sinh\left(\frac{h - x_2}{a}\right) \end{aligned}$$

and using these equations we can write

$$\begin{aligned} v &= a\cosh\left(\frac{x_2}{a}\right) - a\cosh\left(\frac{h - x_2}{a}\right) \\ L &= a\sinh\left(\frac{x_2}{a}\right) + a\sinh\left(\frac{h - x_2}{a}\right) \end{aligned} \tag{43}$$

and dividing one equation by the other gives

$$\frac{v}{L} = \frac{\cosh\left(\frac{x_2}{a}\right) - \cosh\left(\frac{h - x_2}{a}\right)}{\sinh\left(\frac{x_2}{a}\right) + \sinh\left(\frac{h - x_2}{a}\right)}$$

Using the hyperbolic identities $\cosh A - \cosh B = 2\sinh\left(\frac{A+B}{2}\right)\sinh\left(\frac{A-B}{2}\right)$ and

$\sinh A + \sinh B = 2\sinh\left(\frac{A+B}{2}\right)\cosh\left(\frac{A-B}{2}\right)$ gives $\frac{v}{L} = \dfrac{2\sinh\left(\frac{h}{2a}\right)\sinh\left(\frac{2x_2 - h}{2a}\right)}{2\sinh\left(\frac{h}{2a}\right)\cosh\left(\frac{2x_2 - h}{2a}\right)}$ and

$$\frac{v}{L} = \tanh\left(\frac{2x_2 - h}{2a}\right)$$

and solving this equation for x_2/a gives

$$\frac{x_2}{a} = \frac{h}{2a} + \tanh^{-1}\left(\frac{v}{L}\right) \qquad \text{or} \qquad x_2 = \frac{h}{2} + a\tanh^{-1}\left(\frac{v}{L}\right) \tag{44}$$

We can now substitute (44) into the second member of (43) giving (Cella 1999, eq. 7)

$$L = a\sinh\left(\frac{h}{2a} + \tanh^{-1}\left(\frac{v}{L}\right)\right) + a\sinh\left(\frac{h}{2a} - \tanh^{-1}\left(\frac{v}{L}\right)\right) \tag{45}$$

and denoting $q = \tanh^{-1}\left(\frac{v}{L}\right)$ we have

$$L = a\left\{\sinh\left(\frac{h}{2a} + q\right) + \sinh\left(\frac{h}{2a} - q\right)\right\} \tag{46}$$

As before, this equation cannot be solved algebraically for the parameter a, instead a solution using Newton-Raphson iteration is a practical alternative. For this numerical technique (see above) we have

$$a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)} \tag{47}$$

where, with $q = \tanh^{-1}\left(\frac{v}{L}\right)$

$$f(a) = a\left\{\sinh\left(\frac{h}{2a} + q\right) + \sinh\left(\frac{h}{2a} - q\right)\right\} - L \tag{48}$$

And the derivative $f'(a) = \frac{d}{da}\{f(a)\}$ is

$$\begin{aligned} f'(a) &= \sinh\left(\frac{h}{2a} + q\right) - \frac{h}{2a}\cosh\left(\frac{h}{2a} + q\right) + \sinh\left(\frac{h}{2a} - q\right) - \frac{h}{2a}\cosh\left(\frac{h}{2a} - q\right) \\ &= \sinh\left(\frac{h}{2a} + q\right) + \sinh\left(\frac{h}{2a} - q\right) - \frac{h}{2a}\left\{\cosh\left(\frac{h}{2a} + q\right) + \cosh\left(\frac{h}{2a} - q\right)\right\} \end{aligned}$$

Using the hyperbolic identities $\sinh A + \sinh B = 2\sinh\left(\frac{A+B}{2}\right)\cosh\left(\frac{A-B}{2}\right)$ and

$\cosh A + \cosh B = 2\cosh\left(\frac{A+B}{2}\right)\cosh\left(\frac{A-B}{2}\right)$ gives

$$f'(a) = 2\sinh\left(\frac{h}{2a}\right)\cosh q - \frac{h}{2a}\left\{2\cosh\left(\frac{h}{2a}\right)\cosh q\right\} = 2\cosh q\left\{\sinh\left(\frac{h}{2a}\right) - \frac{h}{2a}\cosh\left(\frac{h}{2a}\right)\right\} \tag{49}$$

Similarly to the previous method, after solving for a, x_2 can be found from (44) then $x_1 = x_2 - h$ and y_1, y_2 and s_1, s_2 follow from (9) and (10).

For the example above, a table of values for Newton-Raphson iteration (47), (48) and (49) with $h = 100$, $v = 20$ and $L = 110$ is shown below.

$$a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)} \quad \text{where} \quad \begin{cases} f(a) = a\left\{\sinh\left(\dfrac{h}{2a} + q\right) + \sinh\left(\dfrac{h}{2a} - q\right)\right\} - L \\ f'(a) = 2\cosh q\left\{\sinh\left(\dfrac{h}{2a}\right) - \dfrac{h}{2a}\cosh\left(\dfrac{h}{2a}\right)\right\} \end{cases} \quad \text{and } q = \tanh^{-1}\left(\frac{v}{L}\right)$$

n	a_n	$h/(2a)+q$	$h/(2a)-q$	$f(a)$	$f'(a)$	$c = f(a)/f'(a)$	$a_{n+1} = a_n - c$
1	50	1.183862390	0.816137610	9.512127665	-0.748230260	-12.712834768	62.712834768
2	62.712834768	0.981147351	0.613422571	2.816662582	-0.365939796	-7.697065504	70.409900272
3	70.409900272	0.893989798	0.526265018	0.460294480	-0.255248093	-1.803321916	72.213222188
4	72.213222188	0.876256366	0.508531586	0.017641378	-0.236019682	-0.074745367	72.287967555
5	72.287967555	0.875540434	0.507815654	0.000028213	-0.235265308	-0.000119921	72.288087476
6	72.288087476	0.875539287	0.507814506	0.000000000	-0.235264101	0.000000000	72.288087476

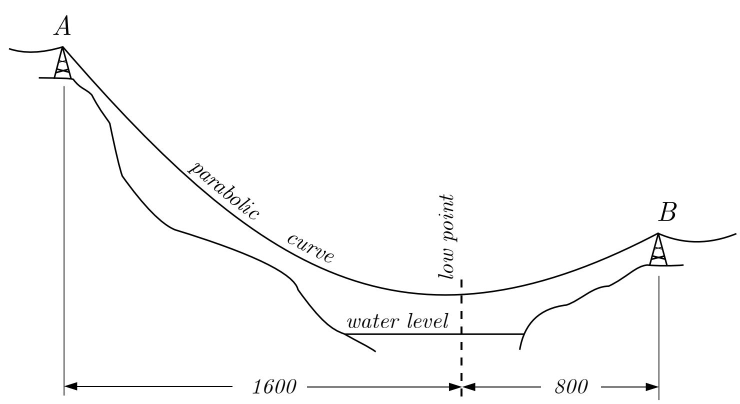
Table 2. Newton-Raphson iteration

The result $a = 72.288087476$ after 6 iterations is identical to a in the previous iterative scheme that used a different function $f(a)$.

The catenary and the parabolic curve: Curly's Conundrum No. 15

Curly's Conundrum No. 15

A power transmission cable, spanning a lake, hangs in a parabolic arc between pylons A and B. The pylons are 2.4 km apart and the lowest point of the cable is 1.6 km from A. If the RL's[4] of the tops of the pylons A and B are 500.0 m and 423.2 m respectively, calculate the minimum clearance if the water level of the lake is 382.0 m.



[4] Reduced Levels (elevations above a datum)

The Institution of Surveyors Victoria (ISV) has a news bulletin *Traverse* that is published quarterly and circulated to members. In *Traverse 321* (March 2019) the puzzle (above) was published as *Curly's Conundrum No. 15.* The solution was published in *Traverse 322* (June 2019) and is summarized below:

Solution to Curly's Conundrum No. 15

1 The power cable hangs in a parabolic curve $y = ax^2 + bx + c$ where the x,y system has its origin at A

2 At A: $y_A = 0,\ x_A = 0$ hence $c = 0$

At B: $y_B = -76.8,\ x_B = 2400$ and $y_B = ax_B^2 + bx_B \quad (\text{i})$

At C: gradient $\dfrac{dy}{dx} = 2ax_C + b = 0 \quad (\text{ii})$

3 Equations (i) and (ii) can be written as a matrix equation $\begin{bmatrix} x_B^2 & x_B \\ 2x_C & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_B \\ 0 \end{bmatrix}$ that can be solved for a and b, where, in matrix notation: $\begin{bmatrix} a \\ b \end{bmatrix} = \dfrac{1}{x_B^2 - 2x_B x_C}\begin{bmatrix} 1 & -x_B \\ -2x_C & x_B^2 \end{bmatrix}\begin{bmatrix} y_B \\ 0 \end{bmatrix} = \begin{bmatrix} 0.000040 \\ -0.128000 \end{bmatrix}$

4 $RL_C = RL_A + y_C = 500.0 + \left(a(1600)^2 + b(1600) + 0\right) = 397.6$

5 Minimum clearance is $397.6 - 382.0 = 15.6$ m

Catenary v's Parabola

Of course, we know that the power cable will not hang in a parabolic arc between the supports, but instead will hang in a catenary. How close are the two curves?

To make a comparison, we need to know the length of cable in the parabolic curve in Curly's Conundrum No. 15 and from Appendix B, the formula for arc length of a parabolic curve $y = ax^2 + bc + c$ from $x = 0$ to $x = t$ is

$$s = \frac{1}{4a}\left\{\begin{array}{l} (2at+b)\sqrt{1+(2at+b)^2} + \ln\left(\sqrt{1+(2at+b)^2} + 2at + b\right) \\ -\left(b\sqrt{1+b^2} + \ln\left(\sqrt{1+b^2} + b\right)\right) \end{array}\right\} \tag{50}$$

So for $a = 0.000040,\ b = -0.128000,\ c = 0$ and $t = 2400$ we have $(2at+b)^2 = (0.064)^2 = 0.004096$ and $1 + b^2 = 1.016384000$ giving

$$\begin{aligned} s &= \frac{1}{4(0.000040)}\left\{\begin{array}{l} 0.064\sqrt{1.004096} + \ln\left(\sqrt{1.004096} + 0.064\right) \\ -\left(-0.128\sqrt{1.016384} + \ln\left(\sqrt{1.016384} + (-0.128)\right)\right) \end{array}\right\} \\ &= 6250\left\{0.128087328 - (-0.256697343)\right\} \\ &= 2404.90418987 \text{ m} \end{aligned}$$

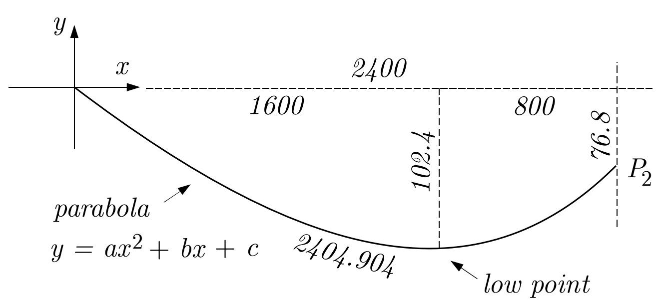


Figure 5. The parabolic arc in Curly's Conundrum No. 15

We can now use this value (rounded to nearest mm) as the length of a catenary $L = 2404.904$ m between supports P_1 and P_2 separated by $h = 2400$ m and $v = -76.8$ m to determine the parameter a in the catenary equation. As before, the solution for a is iterative and the other values of the catenary are set out below

1 Solve for a using Newton-Raphson iteration (38), (39) and (40) with $h = 2400$, $v = -76.8$ and $L = 2404.904$

$$a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)} \quad \text{where} \quad \begin{aligned} f(a) &= 2a\sinh\left(\frac{h}{2a}\right) - \sqrt{L^2 - v^2} \\ f'(a) &= 2\left\{\sinh\left(\frac{h}{2a}\right) - \frac{h}{2a}\cosh\left(\frac{h}{2a}\right)\right\} \end{aligned}$$

A starting value for a can be calculated from (42) as

$$a_1 = \frac{h}{\sqrt{24}}\sqrt{\frac{h}{\sqrt{L^2 - v^2} - h}} = 12515 \text{ (nearest whole number)}$$

n	a_n	$f(a)$	$f'(a)$	$c = f(a)/f'(a)$	$a_{n+1} = a_n - c$
1	12515	0.001866558	-0.000588246	-3.173088943	12518.173088943
2	12518.173088943	0.000000710	-0.000587799	-0.001208021	12518.174296964
3	12518.174296964	0.000000000	-0.000587799	0.000000000	12518.174296964

Table 3. Values for Newton-Raphson iteration

The solution $a = 12518.174296964$ is achieved after 3 iterations using a starting value $a_1 = 12515$

2 Solve for x_2 using (41)

$$x_2 = \frac{h}{2} + a\tanh^{-1}\left(\frac{v}{L}\right) = 800.099294777$$

3 Solve for x_1 using (35) then y_1, y_2 and s_1, s_2 using (9) and (10) respectively

$$\begin{aligned} x_1 &= x_2 - h = -1599.900705223 \\ y_1 &= a\cosh\left(\frac{x_1}{a}\right) = 12620.552181711, \qquad y_2 = a\cosh\left(\frac{x_2}{a}\right) = 12543.752181711 \\ s_1 &= a\sinh\left(\frac{x_1}{a}\right) = -1604.259842457 \qquad s_2 = a\sinh\left(\frac{x_2}{a}\right) = 800.644157543 \end{aligned}$$

4 The sag of the cable below P_1 is the vertical distance $y_1 - a = 102.377884747$

These values are shown (rounded to nearest mm) in Figure 6.

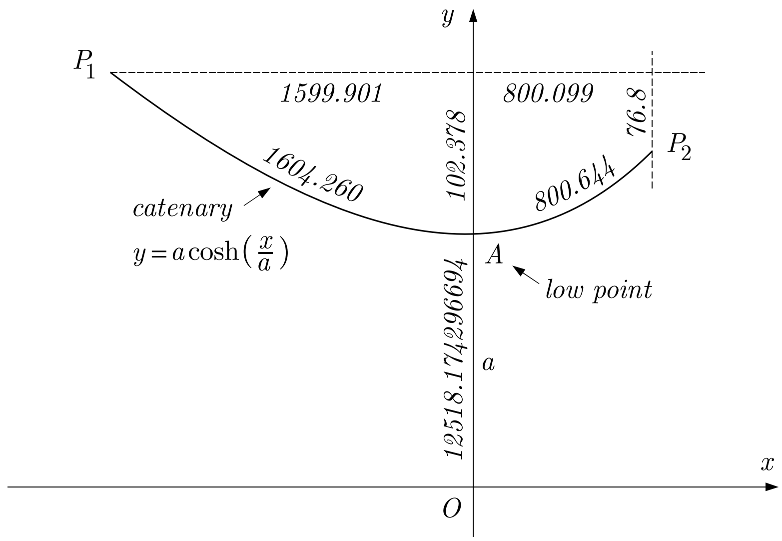


Figure 6. The catenary for Curly's Conundrum No. 15

The following table shows the sag[5] for a parabola and the catenary and the difference between the two. If the difference is positive the parabola is above the catenary. If the difference is negative then the parabola is below the catenary.

Parabola		**Catenary**		
x	y (sag)	horiz dist from P_1	sag related to P_1	difference
0	0.000	0	0.000	0.000
200	-24.000	200	-24.021	0.021
400	-44.800	400	-44.827	0.027
600	-62.400	600	-62.423	0.023
800	-76.800	800	-76.813	0.013
1000	-88.000	1000	-88.001	0.001
1200	-96.000	1200	-95.990	-0.010
1400	-100.800	1400	-100.782	-0.018
1599.900705	-102.400	1599.900705	-102.377885**	-0.022
1600	-102.400000*	1600	-102.378	-0.022
1800	-100.800	1800	-100.779	-0.021
2000	-96.000	2000	-95.983	-0.017
2200	-88.000	2200	-87.991	-0.009
2400	-76.800	2400	-76.800	0.000

Table 4. Differences in sag between Parabola and Catenary for Curly's Conundrum No. 15
(*) and (**) are low points of parabola and catenary respectively

[5] Sag is the vertical distance between the support (P_1 or P_2) and the curve (parabola or catenary).

Conclusion

The catenary is a curve with a relatively simple equation that was discovered by scientists and mathematicians of the 16th and 17th centuries wishing to understand the physics of motion of bodies or the shapes of flexible objects under the action of gravity. This paper has given some history of the catenary's discovery and some of the great mathematicians involved; Galileo, Huygens, Leibniz, Jacob and Daniel Bernoulli prominent amongst many. Two modern methods of derivation of the catenary equations have been provided and the geometry of the curve explained. In addition, examples of calculations are shown as well as a comparison between a catenary and a parabolic arc for a hanging cable. The reference list provides further reading on the topic and appendices show details of associated topics (hyperbolic functions and length of parabolic arc) as well as some extracts from historical writings on the topic.

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Appendix A: Hyperbolic functions

The basic functions are the hyperbolic sine of x, denoted by $\sinh x$, and the hyperbolic cosine of x denoted by $\cosh x$; they are defined as

$$\begin{aligned} \sinh x &= \frac{e^x - e^{-x}}{2} = \frac{e^{2x} - 1}{2e^x} = \frac{1 - e^{-2x}}{2e^{-x}} \\ \cosh x &= \frac{e^x + e^{-x}}{2} = \frac{e^{2x} + 1}{2e^x} = \frac{1 + e^{-2x}}{2e^{-x}} \end{aligned} \tag{51}$$

Other hyperbolic functions are in terms of these

$$\begin{aligned} \tanh x &= \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}, \quad \coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{e^{2x} + 1}{e^{2x} - 1} \\ \operatorname{sech} x &= \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} = \frac{2e^x}{e^{2x} + 1}, \quad \operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}} = \frac{2e^x}{e^{2x} - 1} \end{aligned} \tag{52}$$

And

$$\cosh^2 x - \sinh^2 x = 1, \quad \operatorname{sech}^2 x + \tanh^2 x = 1, \quad \coth^2 x - \operatorname{csch}^2 x = 1 \tag{53}$$

The inverse hyperbolic function of $\sinh x$ is $\sinh^{-1} x$ and is defined by $\sinh^{-1}\left(\sinh x\right) = x$. Similarly $\cosh^{-1} x$ and $\tanh^{-1} x$ are defined by $\cosh^{-1}\left(\cosh x\right) = x$ and $\tanh^{-1}\left(\tanh x\right) = x$; both requiring $x > 0$ and as a consequence of the definitions (see, for example, Fitch 2016)

$$\begin{aligned} \sinh^{-1} x &= \ln\left(x + \sqrt{x^2 + 1}\right) \qquad -\infty < x < \infty \\ \cosh^{-1} x &= \ln\left(x + \sqrt{x^2 - 1}\right) \qquad x \geq 1 \\ \tanh^{-1} x &= \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right) \qquad -1 < x < 1 \end{aligned} \tag{54}$$

A useful identity linking circular and hyperbolic functions is obtained by considering the following.

Using the trigonometric addition and double angle formula we have

$$\ln\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}x\right) = \ln\frac{\cos\frac{1}{2}x + \sin\frac{1}{2}x}{\cos\frac{1}{2}x - \sin\frac{1}{2}x} = \ln\frac{\left(\cos\frac{1}{2}x + \sin\frac{1}{2}x\right)^2}{\cos^2\frac{1}{2}x - \sin^2\frac{1}{2}x} = \ln\frac{1 + \sin x}{\cos x} \tag{55}$$

Also, replacing x with $\tan x$ in the definition of the inverse hyperbolic functions in equations (54) we have

$$\sinh^{-1}\tan x = \ln\left(\tan x + \sqrt{1 + \tan^2 x}\right) = \ln\left(\tan x + \sec x\right) = \ln\frac{1 + \sin x}{\cos x} \tag{56}$$

And equating $\ln\frac{1 + \sin x}{\cos x}$ from equations (55) and (56) gives

$$\ln\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}x\right) = \ln\left(\tan x + \sec x\right) = \sinh^{-1}\tan x \tag{57}$$

Appendix B: Arc Length of a Parabolic Curve

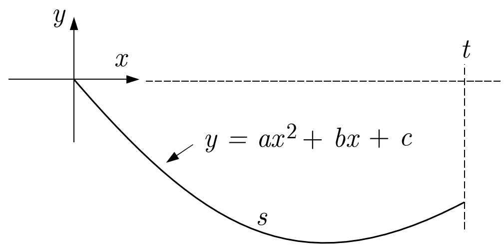

Figure B1. Parabolic curve of length s

The differential arc length $ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1+\left(\frac{dy}{dx}\right)^2}$ and integration gives the arc length $s$of a curve as

$$s = \int_0^t \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \tag{58}$$

And for a parabolic curve $y = ax^2 + bx + c$. $\frac{dy}{dx} = 2ax + b$ then the parabolic arc length is

$$s = \int_0^t \sqrt{1+\left(2ax+b\right)^2}\, dx \tag{59}$$

Let $2ax + b = \tan u$ then $\left(2ax+b\right)^2 = \tan^2 u$ and $1+\left(2ax+b\right)^2 = 1+\tan^2 u = \sec^2 u$. Also $\frac{d}{dx}\left(2ax+b\right) = \frac{d}{du}\left(\tan u\right)\frac{du}{dx}$ and $dx = \frac{1}{2a}\sec^2 u\ du$ and using these results in (59) gives

$$s = \int_0^t \sqrt{1+\left(2ax+b\right)^2}\, dx = \int \sec u \tfrac{1}{2a}\sec^2 u\ du = \frac{1}{2a}\int \sec^3 u\ du$$

Using standard integral results

$$\begin{aligned} s &= \tfrac{1}{2a}\left[\tfrac{1}{2}\sec u \tan u + \tfrac{1}{2}\ln\left(\sec u + \tan u\right)\right] \\ &= \tfrac{1}{2a}\left[\tfrac{1}{2}\left(2ax+b\right)\sqrt{1+\left(2ax+b\right)^2} + \tfrac{1}{2}\ln\left(\sqrt{1+\left(2ax+b\right)^2} + 2ax+b\right)\right]_{x=0}^{x=t} \\ &= \tfrac{1}{4a}\left[\left(2ax+b\right)\sqrt{1+\left(2ax+b\right)^2} + \ln\left(\sqrt{1+\left(2ax+b\right)^2} + 2ax+b\right)\right]_0^t \end{aligned}$$

And the arc length of a parabolic $y = ax^2 + bc + c$ curve from $x = 0$ to $x = t$ is

$$s = \frac{1}{4a}\left\{\begin{array}{l} \left(2at+b\right)\sqrt{1+\left(2at+b\right)^2} + \ln\left(\sqrt{1+\left(2at+b\right)^2} + 2at+b\right) \\ -\left(b\sqrt{1+b^2} + \ln\left(\sqrt{1+b^2} + b\right)\right) \end{array}\right\} \tag{60}$$

Appendix C: Extracts from Galileo's *Discorsi* and *Acta Eruditorum* (1690)

A translation of *Discorsi*

DIALOGUES

CONCERNING

TWO NEW SCIENCES

BY

GALILEO GALILEI

Translated from the Italian and Latin into English by
HENRY CREW AND ALFONSO DE SALVIO
of Northwestern University

WITH AN INTRODUCTION BY
ANTONIO FAVARO
of the University of Padua.

"I think with your friend that it has been of late too much the mode to slight the learning of the ancients."
Benjamin Franklin, *Phil. Trans.* *64*, 445. (1774.)

New York
THE MACMILLAN COMPANY
1914

The following pages (pp.148-149) are a discussion between Sagredo (SAGR.) and Salviati (SALV.) two of the three people (Simplicio is not present here) that Galileo used to put forward explanations, questions and theories on various topic of science.

many propositions. This same theorem has also been recently established by Luca Valerio,* the Archimedes of our age; his demonstration is to be found in his book dealing with the centers of gravity of solids.

SALV. A book which, indeed, is not to be placed second to any produced by the most eminent geometers either of the present or of the past; a book which, as soon as it fell into the hands of our Academician, led him to abandon his own researches along these lines; for he saw how happily everything had been treated and demonstrated by Valerio.

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SAGR. When I was informed of this event by the Academician himself, I begged of him to show the demonstrations which he had discovered before seeing Valerio's book; but in this I did not succeed.

SALV. I have a copy of them and will show them to you; for you will enjoy the diversity of method employed by these two authors in reaching and proving the same conclusions; you will also find that some of these conclusions are explained in different ways, although both are in fact equally correct.

SAGR. I shall be much pleased to see them and will consider it a great favor if you will bring them to our regular meeting. But in the meantime, considering the strength of a solid formed from a prism by means of a parabolic section, would it not, in view of the fact that this result promises to be both interesting and useful in many mechanical operations, be a fine thing if you were to give some quick and easy rule by which a mechanician might draw a parabola upon a plane surface?

SALV. There are many ways of tracing these curves; I will mention merely the two which are the quickest of all. One of these is really remarkable; because by it I can trace thirty or forty parabolic curves with no less neatness and precision, and in a shorter time than another man can, by the aid of a compass, neatly draw four or six circles of different sizes upon paper. I take a perfectly round brass ball about the size of a walnut and project it along the surface of a metallic mirror held

* An eminent Italian mathematician, contemporary with Galileo. [*Trans.*]

in a nearly upright position, so that the ball in its motion will press slightly upon the mirror and trace out a fine sharp parabolic line; this parabola will grow longer and narrower as the angle of elevation increases. The above experiment furnishes clear and tangible evidence that the path of a projectile is a parabola; a fact first observed by our friend and demonstrated by him in his book on motion which we shall take up at our next meeting. In the execution of this method, it is advisable to slightly heat and moisten the ball by rolling in the hand in order that its trace upon the mirror may be more distinct.

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The other method of drawing the desired curve upon the face of the prism is the following: Drive two nails into a wall at a convenient height and at the same level; make the distance between these nails twice the width of the rectangle upon which it is desired to trace the semiparabola. Over these two nails hang a light chain of such a length that the depth of its sag is equal to the length of the prism. This chain will assume the form of a parabola,* so that if this form be marked by points on the wall we shall have described a complete parabola which can be divided into two equal parts by drawing a vertical line through a point midway between the two nails. The transfer of this curve to the two opposing faces of the prism is a matter of no difficulty; any ordinary mechanic will know how to do it.

By use of the geometrical lines drawn upon our friend's compass,† one may easily lay off those points which will locate this same curve upon the same face of the prism.

Hitherto we have demonstrated numerous conclusions pertaining to the resistance which solids offer to fracture. As a starting point for this science, we assumed that the resistance offered by the solid to a straight-away pull was known; from this base one might proceed to the discovery of many other results and their demonstrations; of these results the number to

* It is now well known that this curve is not a parabola but a catenary the equation of which was first given, 49 years after Galileo's death, by James Bernoulli. [*Trans.*]

† The geometrical and military compass of Galileo, described in Nat. Ed. Vol. 2. [*Trans.*]

Acta Eruditorum

The first monograph on the famous learned journal and scientific periodical Acta Eruditorum, *the German counterpart of the* Journal des Sçavans *and the* Philosophical Transactions:

(A.) HUB. LAEVEN

THE "ACTA ERUDITORUM" UNDER THE EDITORSHIP OF OTTO MENCKE

The history of an international learned journal between 1682 and 1707

Translated from the Dutch by Lynne Richards. With a summary in German

With a list of all – hitherto anonymous – contributors, an inventory of the editor's correspondence, a bibliography and an index of names

From 1682, for precisely a century, the Latin *Acta Eruditorum*, one of the most important international journals in the period of the Enlightenment, was published in Leipzig. The founder and first editor-in-chief was Otto Mencke (1644-1707), professor of philosophy at the university of Leipzig. This authoritative journal provided chiefly mathematicians, physicists and other natural scientists, among them Leibniz, with an international platform on which they could present their latest discoveries and ideas. The journal also provides a survey of numerous interesting publications in all other areas of scholarschip.

In this study, which is the result of the most thorough research into the *Acta Eruditorum* undertaken to date, the author examines the history of the journal during the first twenty-five years of its existence, when Otto Mencke was the editor-in-chief. The reader is given detailed information about the genesis of the periodical, and about the members of the editorial circle and the way in which they set about their work. The editorial policy is analyzed in depth, and there is a survey of the international network of correspondents on whom the editors could call. The discovery of a number of annotated copies of the journal has made it possible to establish the names of all the hitherto anonymous reviewers and authors of articles. Several appendices, including an index of contributors and an inventory of the first editor's correspondence, complete the book.

* * *

Catenary Curve

From *Acta Eruditorum,* May 1690, bottom of p. 217

J.B. ANALYSIS PROBLEMATIS ANTEHAC PROPOSITI, de inventione lineæ descensus a corpore gravi percurrendæ uniformiter, sic ut temporibus æqualibus æquales altitudines emetiatur: & alterius cujusdam Problematis Propositio.

Ec Solu-

The "J.B." is Jacob Bernoulli

ANALYSIS PROBLEMATIS ANTEHAC PROPOSITI,
de invention linea descensus a corpore gravi percurrendae uniformiter, sic ut temporibus aequalibus aequales altitudines emetiatur: and alterius cujusdam Problematis Propesitio

Acta Eruditorum, May 1690, top of p. 218

218 ACTA ERUDITORUM

Solutionem Problematis nudam dedit Ill. Hugenius in Nov. Roterod. Hanc postea excepit in Act. Lips. A. 1689. p 195. sqq. celeb. Auctoris Demonstratio Synthetica. Analysin, quam suppressit uterq; ipsius Auctoris calculo differentiali institutam nunc pando, eum in finem, ut Virum Cel. ad par officii genus publico præstandum, tentandamque sua Methodo Problematis deinceps proponendi solutionem invitem.

TAB. VII. *Fig. 1.* Intelligatur grave demissum ab A per curvam quæsitam BFG, in qua sumtæ sint particulæ infinite parvæ, adeoque pro rectis habendæ DG, FH, altitudinum æqualium GI, HL, eæque producantur in M, N, ut fiant Tangentes GM, HN, ipsique HN parallela ducatur GP. Celeritates gravis acquisitæ in G & H eædem sunt, cum iis quas acquireret descendendo perpendiculariter ab eadem linea horizontali AC, per rectas CG, EH, quæ quidem sunt Quadrata ipsarum celeritatum, ut notum. Quibus positis,

Acta Eruditorum, May 1690, bottom of p.219

Problema vicissim proponendum hoc esto:
Invenire, quam curvam referat funis laxus & inter duo puncta fixa libere suspensus. Sumo autem, funem esse lineam in omnibus suis partibus facillime flexilem.

Problema vicissim proponendum hoc esto:
Invenire, quam curvam referat funis laxus & inter duo puncta fixae libere suspensus. Sumo autem, funem esse lineam in omnibus suis partibus facillime flexilem.

Catenary Curve

Some Latin translations:

Problem vicissim	= On the other hand problem
proponendum	= displayed
hoc esto	= to be
Invenire, quam curvam referat funis laxus	= Find the curve resembling a loose cord
inter duo puncta fixae libere suspensus	= freely suspended between two fixed points
Sumo autem	= take it
funem esse lineam	= a rope line
in omnibus suis	= in all
suis partibus	= its parts
facillime flexilem	= easily pliable