TRANSVERSE MERCATOR PROJECTION
Karney-Krueger equations

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The ellipsoidal transverse Mercator (TM) projection is a conformal mapping from the ellipsoid to the plane and is widely used in the geospatial community. The TM projection is also known as the Gauss-Krueger projection acknowledging C.F. Gauss's original development of the ellipsoidal form of the projection and the work of L. Krueger (1912) who re-evaluated both Gauss' work and also the contributions by Oscar Schreiber who used a simplified form of Gauss's projection for the Prussian Land Survey of 1876-1923. Krueger published two sets of equations for the transformations between the ellipsoid and the TM projection; one set – also known as Redfearn's or Thomas's equations, (Redfearn 1948, Thomas 1952) – only accurate within a narrow band of longitude about a central meridian and another more versatile set that offer micrometre accuracy anywhere within 30º of the central meridian. These latter equations, that are far more useful to the geospatial community, have been re-evaluated and improved by Poder & Engsager (1998), Engsager & Poder (2007) and Karney (2011) and are hereinafter described as the *Karney-Krueger equations* to avoid confusion with other sets of TM projection equations.

Deakin *et al.* (2010) also provide a development of the Karney-Krueger equations and show how, in the forward transformation $\phi,\lambda \to E,N$, they represent a triple projection in two parts: the first part is a conformal mapping from the ellipsoid to a sphere (conformal sphere) followed by a conformal mapping from the sphere to the plane using the spherical TM projection equations with spherical latitude ϕ replaced by conformal latitude ϕ'. This two-step process is also known as the Gauss-Schreiber projection and the scale along the central meridian is not constant. The second part is the conformal mapping from the Gauss-Schreiber plane to the TM plane where the scale factor along the central meridian is made constant. This sequence of projections is shown schematically in Figure 1.

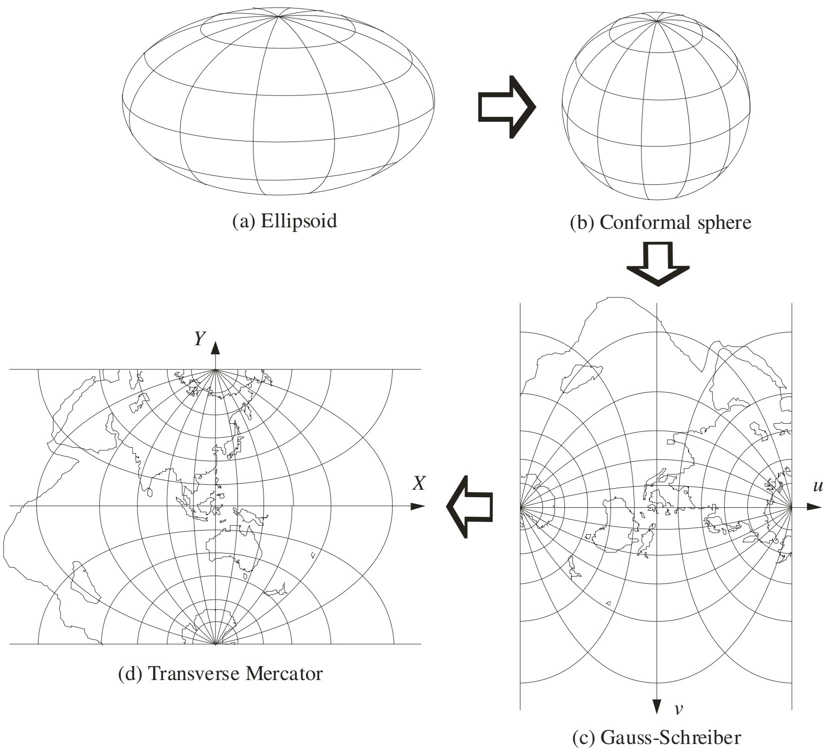

Figure 1 Karney-Krueger equations: sequence of conformal mappings
Ellipsoid → conformal sphere → Gauss-Schreiber plane → transverse Mercator plane.

The following pages set out the terminology and equations necessary for the *Forward* and *Inverse* transformations between geographic coordinates (ϕ,λ) and grid coordinates (E,N) of the transverse Mercator projection of the ellipsoid. The *Forward* transformation (Geographic to Grid) converts latitude and longitude to east and north given the defining parameters of the ellipsoid, the longitude of the central meridian, the scale factor of the central meridian and the offsets of the false origin. The *Inverse* transformation (Grid to Geographic) converts east and north to latitude and longitude. Grid convergence and Point Scale Factor are computed in both transformations.

Nomenclature

α_{2k}	coefficients (Forward transformation)
β_{2k}	coefficients (Inverse transformation)
ε	eccentricity of ellipsoid
ε^2	eccentricity of ellipsoid squared
η	transverse Mercator ratio $\eta = X/A$
η'	Gauss-Schreiber ratio $\eta' = v/a$
λ	longitude
λ_0	longitude of central meridian
ξ	transverse Mercator ratio $\xi = Y/A$
ξ'	Gauss-Schreiber ratio $\xi' = u/a$
σ	function of latitude
ϕ	latitude
ϕ'	conformal latitude
ω	longitude difference: $\omega = \lambda - \lambda_0$
ζ	complex variable: $\zeta = \xi + i\eta$
ζ'	complex variable: $\zeta' = \xi' + i\eta'$
A	rectifying radius
a	semi-major axis of ellipsoid and radius of conformal sphere
d_k^{Re}, d_k^{Im}	numeric variables in Clenshaw summation
E	east grid coordinate
E_0	false origin offset
f	flattening of ellipsoid
f^{Re}, f^{Im}	numeric constants in Clenshaw summation
F^{Re}, F^{Im}	numeric constants in Clenshaw summation
g_k^{Re}, g_k^{Im}	numeric variables in Clenshaw summation
m_0	central meridian scale factor
N	north grid coordinate
N_0	false origin offset
n	3rd flattening of ellipsoid
p, q	numeric variables in scale and grid convergence equations
t	$t = \tan\phi$
t'	$t' = \tan\phi'$
u	Gauss-Schreiber coordinate (north)
v	Gauss-Schreiber coordinate (east)
w_k^{Re}, w_k^{Im}	numeric variables in Clenshaw summation
X	transverse Mercator coordinate (east)
Y	transverse Mercator coordinate (north)

Forward transformation (Geographic to Grid): $\phi,\lambda \rightarrow E,N$ given $a, f, \lambda_0, m_0, E_0, N_0$

1. Compute ellipsoid constants ε^2 and n from

$$\varepsilon^2 = f(2-f) \qquad\qquad n = \frac{f}{2-f} \tag{1}$$

2. Compute the rectifying radius A from

$$A = \frac{a}{1+n}\left\{1+\frac{1}{4}n^2+\frac{1}{64}n^4+\frac{1}{256}n^6+\frac{25}{16384}n^8+\cdots\right\} \tag{2}$$

For efficient numerical evaluation the equation for the rectifying radius should be expressed in Horner form

$$A = \frac{a}{1+n}\left(n^2\left(n^2\left(n^2\left(25n^2+64\right)+256\right)+4096\right)+16384\right)\Big/16384 \tag{3}$$

3. Compute the coefficients $\{\alpha_{2k}\}$ for $k = 1, 2, \ldots, 8$ from

$$
\begin{aligned}
\alpha_2 &= \frac{1}{2}n - \frac{2}{3}n^2 + \frac{5}{16}n^3 + \frac{41}{180}n^4 - \frac{127}{288}n^5 + \frac{7891}{37800}n^6 + \frac{72161}{387072}n^7 - \frac{18975107}{50803200}n^8 \\
\alpha_4 &= \frac{13}{48}n^2 - \frac{3}{5}n^3 + \frac{557}{1440}n^4 + \frac{281}{630}n^5 - \frac{1983433}{1935360}n^6 + \frac{13769}{28800}n^7 + \frac{148003883}{174182400}n^8 \\
\alpha_6 &= \frac{61}{240}n^3 - \frac{103}{140}n^4 + \frac{15061}{26880}n^5 + \frac{167603}{181440}n^6 - \frac{67102379}{29030400}n^7 + \frac{79682431}{79833600}n^8 \\
\alpha_8 &= \frac{49561}{161280}n^4 - \frac{179}{168}n^5 + \frac{6601661}{7257600}n^6 + \frac{97445}{49896}n^7 - \frac{40176129013}{7664025600}n^8 \\
\alpha_{10} &= \frac{34729}{80640}n^5 - \frac{3418889}{1995840}n^6 + \frac{14644087}{9123840}n^7 + \frac{2605413599}{622702080}n^8 \\
\alpha_{12} &= \frac{212378941}{319334400}n^6 - \frac{30705481}{10378368}n^7 + \frac{175214326799}{58118860800}n^8 \\
\alpha_{14} &= \frac{1522256789}{1383782400}n^7 - \frac{16759934899}{3113510400}n^8 \\
\alpha_{16} &= \frac{1424729850961}{743921418240}n^8
\end{aligned}
\tag{4}
$$

For efficient numerical evaluation the coefficients $\{\alpha_{2k}\}$ for $k = 1, 2, \ldots, 8$ should be expressed in Horner form

$$
\begin{aligned}
\alpha_2 &= \Big(n\Big(n\Big(n\Big(n\Big(n\Big(n\big((37884525 - 75900428n)n + 42422016\big) \\
&\quad - 89611200\Big) + 46287360\Big) + 63504000\Big) - 135475200\Big) + 101606400\Big)\Big)\Big/203212800 \\
\alpha_4 &= \Big(n^2\Big(n\Big(n\Big(n\Big(n\Big(n\big(148003883n + 83274912\big) - 178508970\Big) \\
&\quad + 77690880\Big) + 67374720\Big) - 104509440\Big) + 47174400\Big)\Big)\Big/174182400 \\
\alpha_6 &= \Big(n^3\Big(n\Big(n\Big(n\big(n(318729724n - 738126169) + 294981280\big) \\
&\quad + 178924680\Big) - 234938880\Big) + 81164160\Big)\Big)\Big/319334400 \\
\alpha_8 &= \Big(n^4\Big(n\Big(n\big((14967552000 - 40176129013n)n + 6971354016\big) \\
&\quad - 8165836800\Big) + 2355138720\Big)\Big)\Big/7664025600 \\
\alpha_{10} &= \Big(n^5\Big(n\big(n(10421654396n + 3997835751) - 4266773472\big) + 1072709352\Big)\Big)\Big/2490808320 \\
\alpha_{12} &= \Big(n^6\big(n(175214326799n - 171950693600) + 38652967262\big)\Big)\Big/58118860800 \\
\alpha_{14} &= (13700311101 - 67039739596n)n^7\big/12454041600 \\
\alpha_{16} &= 1424729850961n^8\big/743921418240
\end{aligned}
\tag{5}
$$

4. Compute conformal latitude ϕ' from

$$\tan\phi' = \tan\phi\sqrt{1+\sigma^2} - \sigma\sqrt{1+\tan^2\phi} \tag{6}$$

where
$$\sigma = \sinh\left\{\varepsilon \tanh^{-1}\left(\frac{\varepsilon\tan\phi}{\sqrt{1+\tan^2\phi}}\right)\right\} \tag{7}$$

5. Compute longitude difference ω from

$$\omega = \lambda - \lambda_0 \tag{8}$$

6. Compute the Gauss-Schreiber ratios $\xi' = \dfrac{u}{a}$ and $\eta' = \dfrac{v}{a}$ from

$$\xi' = \tan^{-1}\left(\frac{\tan\phi'}{\cos\omega}\right) \qquad \eta' = \sinh^{-1}\left(\frac{\sin\omega}{\sqrt{\tan^2\phi' + \cos^2\omega}}\right) \tag{9}$$

7. Compute transverse Mercator (TM) ratios $\eta = \dfrac{X}{A}$ and $\xi = \dfrac{Y}{A}$ (to order n^8 and $N = 8$) from

$$\eta = \eta' + \sum_{k=1}^{N} \alpha_{2k} \cos 2k\xi' \sinh 2k\eta' \qquad \xi = \xi' + \sum_{k=1}^{N} \alpha_{2k} \sin 2k\xi' \cosh 2k\eta' \tag{10}$$

A very efficient computation of the TM ratios that avoids the need for multiple trigonometric evaluations can be achieved by considering the following.

With the TM ratios $\eta = X/A$, $\xi = Y/A$ and Gauss-Schreiber ratios $\xi' = u/a$, $\eta' = v/a$ the complex function representing the conformal transformation from the *u,v* Gauss-Schreiber plane to the *X,Y* TM plane is $\xi + i\eta = \xi' + i\eta' + \sum_{k=1}^{\infty} \alpha_{2k} \sin 2k\left(\xi' + i\eta'\right)$ (Deakin *et al.* 2012) And with complex variables $\zeta = \xi + i\eta$ and $\zeta' = \xi' + i\eta'$ the transformation can be expressed as (Karney 2011, Deakin & Hunter 2011)

$$\zeta = \zeta' + \sum_{k=1}^{\infty} \alpha_{2k} \sin 2k\zeta' \tag{11}$$

The trigonometric series $\sum_{k=1}^{\infty} \alpha_{2k} \sin 2k\zeta'$ in (11) can be evaluated by Clenshaw summation (see Appendix) which minimizes the number of evaluations of trigonometric and hyperbolic functions leading to

$$\begin{aligned} \eta &= \eta' + g_1^{\mathrm{Im}} \sin 2\xi' \cosh 2\eta' + g_1^{\mathrm{Re}} \cos 2\xi' \sinh 2\eta' \\ \xi &= \xi' + g_1^{\mathrm{Re}} \sin 2\xi' \cosh 2\eta' - g_1^{\mathrm{Im}} \cos 2\xi' \sinh 2\eta' \end{aligned} \tag{12}$$

$g_1^{\mathrm{Re}}, g_1^{\mathrm{Im}}$ are computed from the recurrence relations

$$\begin{aligned} g_k^{\mathrm{Re}} &= \begin{cases} 0, & \text{for } k > N \text{ and } k < 1 \\ 2\left(f^{\mathrm{Re}} g_{k+1}^{\mathrm{Re}} + f^{\mathrm{Im}} g_{k+1}^{\mathrm{Im}}\right) - g_{k+2}^{\mathrm{Re}} + \alpha_{2k}, & \text{for } k = N, N-1, \ldots, 3, 2, 1 \end{cases} \\ g_k^{\mathrm{Im}} &= \begin{cases} 0, & \text{for } k > N \text{ and } k < 1 \\ 2\left(f^{\mathrm{Re}} g_{k+1}^{\mathrm{Im}} - f^{\mathrm{Im}} g_{k+1}^{\mathrm{Re}}\right) - g_{k+2}^{\mathrm{Im}}, & \text{for } k = N, N-1, \ldots, 3, 2, 1 \end{cases} \end{aligned} \tag{13}$$

where

$$f^{\mathrm{Re}} = \cos 2\xi' \cosh 2\eta' \qquad f^{\mathrm{Im}} = \sin 2\xi' \sinh 2\eta' \tag{14}$$

8. Compute *X,Y* transverse Mercator (TM) coordinates

$$X = A\eta \qquad Y = A\xi \tag{15}$$

9. Compute *E,N* grid coordinates

$$E = m_0 X + E_0 \qquad N = m_0 Y + N_0 \tag{16}$$

10. Compute factors *p* and *q* (to order n^8 and $N = 8$) from

$$p = 1 + \sum_{k=1}^{N} 2k\alpha_{2k} \cos 2k\xi' \cosh 2k\eta' \qquad q = -\sum_{k=1}^{N} 2k\alpha_{2k} \sin 2k\xi' \sinh 2k\eta' \tag{17}$$

A very efficient computation of p and q that avoids the need for multiple trigonometric evaluations can be achieved by considering the following.

The conformal transformation from the Gauss-Schreiber plane to the TM plane can be represented by the complex expression $\zeta = \zeta' + \sum_{k=1}^{\infty} \alpha_{2k} \sin 2k\zeta'$ (see (11) above). Differentiating this expression with respect to ζ' and then expressing this derivative as $\dfrac{d\zeta}{d\zeta'} = p + iq$ leads to the complex equation

$$p + iq = 1 + \sum_{k=1}^{\infty} 2k\alpha_{2k} \cos 2k\zeta' \tag{18}$$

The trigonometric series $\sum_{k=1}^{\infty} 2k\alpha_{2k} \cos 2k\zeta'$ in (18) can be evaluated by Clenshaw summation (see Appendix) leading to

$$\begin{aligned} p &= 1 + d_1^{\mathrm{Re}} f^{\mathrm{Re}} + d_1^{\mathrm{Im}} f^{\mathrm{Im}} - d_2^{\mathrm{Re}} \\ q &= \phantom{1 + {}} d_1^{\mathrm{Im}} f^{\mathrm{Re}} - d_1^{\mathrm{Re}} f^{\mathrm{Im}} - d_2^{\mathrm{Im}} \end{aligned} \tag{19}$$

$d_1^{\mathrm{Re}}, d_1^{\mathrm{Im}}, d_2^{\mathrm{Re}}, d_2^{\mathrm{Im}}$ are computed from the recurrence relations

$$\begin{aligned} d_k^{\mathrm{Re}} &= \begin{cases} 0, & \text{for } k > N \text{ and } k < 1 \\ 2\left(f^{\mathrm{Re}} d_{k+1}^{\mathrm{Re}} + f^{\mathrm{Im}} d_{k+1}^{\mathrm{Im}}\right) - d_{k+2}^{\mathrm{Re}} + 2k\alpha_{2k}, & \text{for } k = N, N-1, \ldots, 3, 2, 1 \end{cases} \\ d_k^{\mathrm{Im}} &= \begin{cases} 0, & \text{for } k > N \text{ and } k < 1 \\ 2\left(f^{\mathrm{Re}} d_{k+1}^{\mathrm{Im}} - f^{\mathrm{Im}} d_{k+1}^{\mathrm{Re}}\right) - d_{k+2}^{\mathrm{Im}}, & \text{for } k = N, N-1, \ldots, 3, 2, 1 \end{cases} \end{aligned} \tag{20}$$

11. Compute point scale factor m from

$$m = m_0 \left(\frac{A}{a}\right) \sqrt{p^2 + q^2} \left\{ \frac{\sqrt{1 + \tan^2 \phi}\sqrt{1 - \varepsilon^2 \sin^2 \phi}}{\sqrt{\tan^2 \phi' + \cos^2 \omega}} \right\} \tag{21}$$

12. Compute grid convergence γ from

$$\gamma = \tan^{-1}\left\{ \left| \frac{q}{p} \right| \right\} + \tan^{-1}\left\{ \frac{\left|\tan \phi' \tan \omega\right|}{\sqrt{1 + \tan^2 \phi'}} \right\} \tag{22}$$

Inverse transformation (Grid to Geographic): $E, N \rightarrow \phi, \lambda$ given $a, f, \lambda_0, m_0, E_0, N_0$

1. Compute ellipsoid constants ε^2 and *n*. See *Forward Transformation,* Section 1, equations (1)
2. Compute the rectifying radius *A*. See *Forward Transformation,* Section 2, equation (3)
3. Compute the coefficients $\{\alpha_{2k}\}$ for $k = 1, 2, \ldots, 8$. See *Forward Transformation,* Section 3, equations (5)
4. Compute the coefficients $\{\beta_{2k}\}$ for $k = 1, 2, \ldots, 8$ from

$$\begin{aligned}
\beta_2 &= -\frac{1}{2}n + \frac{2}{3}n^2 - \frac{37}{96}n^3 + \frac{1}{360}n^4 + \frac{81}{512}n^5 - \frac{96199}{604800}n^6 + \frac{5406467}{38707200}n^7 - \frac{7944359}{67737600}n^8 \\
\beta_4 &= -\frac{1}{48}n^2 - \frac{1}{15}n^3 + \frac{437}{1440}n^4 - \frac{46}{105}n^5 + \frac{1118711}{3870720}n^6 - \frac{51841}{1209600}n^7 - \frac{24749483}{348364800}n^8 \\
\beta_6 &= -\frac{17}{480}n^3 + \frac{37}{840}n^4 + \frac{209}{4480}n^5 - \frac{5569}{90720}n^6 - \frac{9261899}{58060800}n^7 + \frac{6457463}{17740800}n^8 \\
\beta_8 &= -\frac{4397}{161280}n^4 + \frac{11}{504}n^5 + \frac{830251}{7257600}n^6 - \frac{466511}{2494800}n^7 - \frac{324154477}{7664025600}n^8 \\
\beta_{10} &= -\frac{4583}{161280}n^5 + \frac{108847}{3991680}n^6 + \frac{8005831}{63866880}n^7 - \frac{22894433}{124540416}n^8 \\
\beta_{12} &= -\frac{20648693}{638668800}n^6 + \frac{16363163}{518918400}n^7 + \frac{2204645983}{12915302400}n^8 \\
\beta_{14} &= -\frac{219941297}{5535129600}n^7 + \frac{497323811}{12454041600}n^8 \\
\beta_{16} &= -\frac{191773887257}{3719607091200}n^8
\end{aligned} \tag{23}$$

For efficient numerical evaluation the coefficients $\{\beta_{2k}\}$ for $k = 1, 2, \ldots, 8$ should be expressed in Horner form

$$\begin{aligned}
\beta_2 &= \Bigl(n\Bigl(n\Bigl(n\Bigl(n\Bigl(n\Bigl(n\bigl((37845269 - 31777436n)n - 43097152\bigr) \\
&\quad + 42865200\Bigr) + 752640\Bigr) - 104428800\Bigr) + 180633600\Bigr) - 135475200\Bigr)\Bigr)\Big/270950400 \\
\beta_4 &= \Bigl(n^2\Bigl(n\Bigl(n\Bigl(n\Bigl(n\bigl((-24749483n - 14930208)n + 100683990\bigr) \\
&\quad - 152616960\Bigr) + 105719040\Bigr) - 23224320\Bigr) - 7257600\Bigr)\Bigr)\Big/348364800 \\
\beta_6 &= \Bigl(n^3\Bigl(n\Bigl(n\Bigl(n\bigl(n(232468668n - 101880889) - 39205760\bigr) \\
&\quad + 29795040\Bigr) + 28131840\Bigr) - 22619520\Bigr)\Bigr)\Big/638668800 \\
\beta_8 &= \Bigl(n^4\Bigl(n\Bigl(n\bigl((-324154477n - 1433121792)n + 876745056\bigr) \\
&\quad + 167270400\Bigr) - 208945440\Bigr)\Bigr)\Big/7664025600 \\
\beta_{10} &= n^5\Bigl(n\bigl((312227409 - 457888660n)n + 67920528\bigr) - 70779852\Bigr)\Big/2490808320 \\
\beta_{12} &= n^6\bigl(n(19841813847n + 3665348512) - 3758062126\bigr)\big/116237721600 \\
\beta_{14} &= n^7(1989295244n - 1979471673)\big/49816166400 \\
\beta_{16} &= -191773887257n^8\big/3719607091200
\end{aligned} \tag{24}$$

5. Compute the transverse Mercator X,Y coordinates from

$$X = \frac{E - E_0}{m_0} \qquad Y = \frac{N - N_0}{m_0} \tag{25}$$

6. Compute the transverse Mercator (TM) ratios ξ and η from

$$\xi = \frac{Y}{A} \qquad \eta = \frac{X}{A} \tag{26}$$

7. Compute the Gauss-Schreiber ratios $\xi' = \dfrac{u}{a}$ and $\eta' = \dfrac{v}{a}$ from

$$\xi' = \xi + \sum_{k=1}^{N} \beta_{2k} \sin 2k\xi \cosh 2k\eta \qquad \eta' = \eta + \sum_{k=1}^{N} \beta_{2k} \cos 2k\xi \sinh 2k\eta \tag{27}$$

A very efficient computation of the Gauss-Schreiber ratios that avoids the need for multiple trigonometric evaluations can be achieved by considering the following.

With the Gauss-Schreiber ratios $\xi' = u/a$, $\eta' = v/a$ and TM ratios $\eta = X/A$, $\xi = Y/A$ the complex function representing the conformal transformation from the X,Y TM plane to the u,v Gauss-Schreiber plane is $\xi' + i\eta' = \xi + i\eta + \sum_{k=1}^{\infty} \beta_{2k} \sin 2k\left(\xi + i\eta\right)$ (Deakin *et al.* 2012) And with complex variables $\zeta = \xi + i\eta$ and $\zeta' = \xi' + i\eta'$ the transformation can be expressed as (Karney 2011, Deakin & Hunter 2011)

$$\zeta' = \zeta + \sum_{k=1}^{\infty} \beta_{2k} \sin 2k\zeta \tag{28}$$

The trigonometric series $\sum_{k=1}^{\infty} \beta_{2k} \sin 2k\zeta$ in (28) can be evaluated by Clenshaw summation (see Appendix) leading to

$$\begin{aligned} \xi' &= \xi + w_1^{\mathrm{Re}} \sin 2\xi \cosh 2\eta - w_1^{\mathrm{Im}} \cos 2\xi \sinh 2\eta \\ \eta' &= \eta + w_1^{\mathrm{Im}} \sin 2\xi \cosh 2\eta + w_1^{\mathrm{Re}} \cos 2\xi \sinh 2\eta \end{aligned} \tag{29}$$

$w_1^{\mathrm{Re}}, w_1^{\mathrm{Im}}$ are computed from the recurrence relations

$$\begin{aligned} w_k^{\mathrm{Re}} &= \begin{cases} 0, & \text{for } k > N \text{ and } k < 1 \\ 2\left(F^{\mathrm{Re}} w_{k+1}^{\mathrm{Re}} + F^{\mathrm{Im}} w_{k+1}^{\mathrm{Im}}\right) - w_{k+2}^{\mathrm{Re}} + \beta_{2k}, & \text{for } k = N, N-1, \ldots, 3, 2, 1 \end{cases} \\ w_k^{\mathrm{Im}} &= \begin{cases} 0, & \text{for } k > N \text{ and } k < 1 \\ 2\left(F^{\mathrm{Re}} w_{k+1}^{\mathrm{Im}} - F^{\mathrm{Im}} w_{k+1}^{\mathrm{Re}}\right) - w_{k+2}^{\mathrm{Re}}, & \text{for } k = N, N-1, \ldots, 3, 2, 1 \end{cases} \end{aligned} \tag{30}$$

where
$$F^{\mathrm{Re}} = \cos 2\xi \cosh 2\eta \qquad F^{\mathrm{Im}} = \sin 2\xi \sinh 2\eta \tag{31}$$

8. Compute $t' = \tan\phi'$ (where ϕ' is conformal latitude) from

$$t' = \tan\phi' = \frac{\sin\xi'}{\sqrt{\sinh^2\eta' + \cos^2\xi'}} \tag{32}$$

9. Solve for $t = \tan\phi$ by Newton-Raphson iteration and then determine latitude ϕ

The equations linking $t = \tan\phi$ and $t' = \tan\phi'$ are (6) and (7) given here in modified form as

$$t' = t\sqrt{1+\sigma^2} - \sigma\sqrt{1+t^2} \tag{33}$$

where
$$\sigma = \sinh\left\{\varepsilon \tanh^{-1}\left(\frac{\varepsilon t}{\sqrt{1+t^2}}\right)\right\} \tag{34}$$

t can be evaluated using the Newton-Raphson method for the real roots of the equation $f(t)=0$ given in the form of an iterative equation

$$t_{n+1}=t_n-\frac{f(t_n)}{f'(t_n)} \tag{35}$$

where t_n denotes the nth iterate and $f(t)$ is given by

$$f(t)=t\sqrt{1+\sigma^2}-\sigma\sqrt{1+t^2}-t' \tag{36}$$

and $t'=\tan\phi'$ is a fixed value. The derivative $f'(t)=\frac{d}{dt}\{f(t)\}$ is given by

$$f'(t)=\left(\sqrt{1+\sigma^2}\sqrt{1+t^2}-\sigma t\right)\frac{\left(1-\varepsilon^2\right)\sqrt{1+t^2}}{1+\left(1-\varepsilon^2\right)t^2} \tag{37}$$

An initial value for t_1 can be taken as $t_1=t'=\tan\phi'$ and the functions $f(t_1)$ and $f'(t_1)$ evaluated from equations (34), (36) and (37). t_2 is now computed from equation (35) and this process repeated to obtain $t_3, t_4, \ldots$. This iterative process can be concluded when the difference between t_{n+1} and t_n reaches an acceptably small value, and then the latitude is given by

$$\phi=\tan^{-1}t_{n+1} \tag{38}$$

10. Compute longitude difference ω and longitude λ from

$$\tan\omega=\frac{\sinh\eta'}{\cos\xi'} \qquad \lambda=\lambda_0\pm\omega \tag{39}$$

11. Compute factors *p* and *q*. See *Forward Transformation,* Section 10, equations (19) and (20).
12. Compute point scale factor *m*. See *Forward Transformation,* Section 11, equation (21).
13. Compute grid convergence γ. See *Forward Transformation,* Section 12, equation (22)

Appendix

Hyperbolic functions

The basic functions are the hyperbolic sine of x, denoted by $\sinh x$, and the hyperbolic cosine of x denoted by $\cosh x$; they are defined as

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2} \tag{40}$$

Other hyperbolic functions are in terms of these

$$\tanh x = \frac{\sinh x}{\cosh x}, \quad \coth x = \frac{1}{\tanh x}, \quad \operatorname{sech} x = \frac{1}{\cosh x}, \quad \operatorname{cosech} x = \frac{1}{\sinh x} \tag{41}$$

The inverse hyperbolic function of $\sinh x$ is $\sinh^{-1} x$ and is defined by $\sinh^{-1}(\sinh x) = x$. Similarly $\cosh^{-1} x$ and $\tanh^{-1} x$ are defined by $\cosh^{-1}(\cosh x) = x$ and $\tanh^{-1}(\tanh x) = x$; both requiring $x > 0$ and as a consequence of the definitions

$$\begin{aligned} \sinh^{-1} x &= \ln\left(x + \sqrt{x^2+1}\right) && -\infty < x < \infty \\ \cosh^{-1} x &= \ln\left(x + \sqrt{x^2-1}\right) && x \ge 1 \\ \tanh^{-1} x &= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) && -1 < x < 1 \end{aligned} \tag{42}$$

Recurrence Relations

A recurrence relation is an equation that recursively defines a sequence. Once one or more initial terms are given each further term of the sequence is defined as a function of the preceding terms. As examples, consider the trigonometric functions

$$\sin k\phi = 2\cos\phi \sin(k-1)\phi - \sin(k-2)\phi \tag{43}$$

$$\cos k\phi = 2\cos\phi \cos(k-1)\phi - \cos(k-2)\phi \tag{44}$$

With initial values $\sin(0) = 0$, $\cos(0) = 1$ in (43) and (44) gives successively

$$\begin{aligned} &\sin 2\phi = 2\cos\phi\sin\phi, && \cos 2\phi = 2\cos^2\phi - 1 \\ &\sin 3\phi = 2\cos\phi\sin 2\phi - \sin\phi, && \cos 3\phi = 2\cos\phi\cos 2\phi - \cos\phi \\ &\sin 4\phi = 2\cos\phi\sin 3\phi - \sin 2\phi, && \cos 4\phi = 2\cos\phi\cos 3\phi - \cos 2\phi \\ &\sin 5\phi = \cdots && \cos 5\phi = \cdots \end{aligned}$$

Recurrence relations for even multiples are obtained by replacing ϕ with 2ϕ in (43) and (44) to give

$$\sin 2k\phi = 2\cos 2\phi \sin 2(k-1)\phi - \sin 2(k-2)\phi \tag{45}$$

$$\cos 2k\phi = 2\cos 2\phi \cos 2(k-1)\phi - \cos 2(k-2)\phi \tag{46}$$

Clenshaw summation

Suppose that a (truncated) sum S is denoted by

$$S = u_0 F_0(x) + u_1 F_1(x) + u_2 F_2(x) + \cdots + u_N F_N(x) = \sum_{k=0}^{N} u_k F_k(x) \tag{47}$$

u_k are coefficients independent of x, and $F(x)$ obey the recurrence relation

$$F_{k+1}(x) = a_k F_k(x) + b_k F_{k-1}(x) \tag{48}$$

where the coefficients a_k, b_k may be functions of x as well as k. Note that in many applications a does not depend on k, and b is a constant independent of x or k.

The sum S can be evaluated from

$$S = b_1 F_0(x)\, y_2 + F_1(x)\, y_1 + F_0(x) u_0 \tag{49}$$

where the quantities y_k are obtained from the 'reverse' recurrence formula

$$y_k = \begin{cases} 0, & \text{for } k > N \\ a_k\, y_{k+1} + b_{k+1}\, y_{k+2} + u_k, & \text{for } k = N, N-1, N-2, \ldots, 3, 2, 1 \end{cases} \tag{50}$$

Equation (50) is Clenshaw's recurrence formula and (49) is the associated sum; equations (49) and (50) combined are called *Clenshaw's summation* (Clenshaw 1955, Deakin & Hunter 2011).

Clenshaw's summation can be explained by writing out (47) as

$$\begin{aligned} S &= u_N F_N(x) + u_{N-1} F_{N-1}(x) + u_{N-2} F_{N-2}(x) + \cdots + u_8 F_8(x) + u_7 F_7(x) + u_6 F_6(x) + \cdots \\ &\quad + u_2 F_2(x) + u_1 F_1(x) + u_0 F_0(x) \end{aligned} \tag{51}$$

and re-arranging (50) as

$$u_k = y_k - a_k\, y_{k+1} - b_k\, y_{k+2} \tag{52}$$

Then substituting (52) into (51) gives

$$\begin{aligned} S &= \left[y_N - a_N\, y_{N+1} - b_{N+1} y_{N+2} \right] F_N(x) + \left[y_{N-1} - a_{N-1} y_N - b_N\, y_{N+1} \right] F_{N-1}(x) \\ &\quad + \left[y_{N-2} - a_{N-2} y_{N-1} - b_{N-1} y_N \right] F_{N-2}(x) + \cdots \\ &\quad + \left[y_8 - a_8 y_9 - b_9 y_{10} \right] F_8(x) + \left[y_7 - a_7 y_8 - b_8 y_9 \right] F_7(x) + \left[y_6 - a_6 y_7 - b_7 y_8 \right] F_6(x) + \cdots \\ &\quad + \left[y_2 - a_2 y_3 - b_3 y_4 \right] F_2(x) + \left[y_1 - a_1 y_2 - b_2 y_3 \right] F_1(x) + \left[u_0 + b_1 y_2 - b_1 y_2 \right] F_0(x) \end{aligned} \tag{53}$$

Noting that in the last line $b_1 y_2$ has been added and subtracted. Examining the terms containing a factor of y_8 in (53) involves

$$\left[F_8(x) - a_7 F_7(x) - b_7 F_6(x) \right] y_8 \tag{54}$$

And as a consequence of the recurrence relation (48) the term in $[\ ]$ will equal zero and similarly for all other y_k down through and including y_2. The only surviving terms in (53) are u_0, y_1 and $b_1 y_2$; and so the sum S is given by (49).

Summation $S = \sum_{k=1}^{N} c_{2k} \sin 2k\phi$

Consider the (truncated) trigonometric series

$$S = c_2 \sin 2\phi + c_4 \sin 4\phi + c_6 \sin 6\phi + \cdots + c_{2N} \sin 2N\phi = \sum_{k=1}^{N} c_{2k} \sin 2k\phi \tag{55}$$

The trigonometric functions $\sin 2\phi, \sin 4\phi, \ldots$ obey the recurrence relation (45) so S can be evaluated using Clenshaw summation. Write the recurrence relation (45) in another form replacing k with $k+1$ giving

$$\sin 2(k+1)\phi = 2\cos 2\phi \sin 2k\phi - \sin 2(k-1)\phi \tag{56}$$

Equation (56) has the same form as (48) where $F_k(x) = \sin 2k\phi$, $a_k = 2\cos 2\phi$ and $b_k = -1$. Clenshaw's recurrence formula (50) becomes

$$y_k = \begin{cases} 0, & \text{for } k > N \\ 2\cos 2\phi\, y_{k+1} - y_{k+2} + c_{2k}, & \text{for } k = N, N-1, N-2, \ldots, 3, 2, 1 \end{cases} \tag{57}$$

The associated sum (see equation (49) with $F_0(x) = \sin(0) = 0$ and $F_1(x) = \sin 2\phi$) is

$$S = \sum_{k=1}^{N} c_{2k} \sin 2k\phi = y_1 \sin 2\phi \tag{58}$$

Complex summation $S=\sum_{k=1}^{N} c_{2k}\sin 2k\zeta$

Consider the complex sum

$$S = S^{\mathrm{Re}} + iS^{\mathrm{Im}} = \sum_{k=1}^{N} c_{2k}\sin 2k\zeta = \sum_{k=1}^{N}\left(c_{2k}+i0\right)\sin 2k\left(\xi+i\eta\right) \tag{59}$$

where S, c_{2k} and ζ are complex numbers having real and imaginary parts. We define a reverse complex recurrence as

$$y_k = \begin{cases} 0, & \text{for } k > N \\ 2\cos 2\zeta\left(y_{k+1}\right) - y_{k+2} + c_{2k}, & \text{for } k = N, N-1, N-2, \ldots, 3, 2, 1 \end{cases} \tag{60}$$

Where $y_k = y_k^{\mathrm{Re}} + iy_k^{\mathrm{Im}}$ are complex numbers having real and imaginary parts and using (58) the complex sum (59) is given as

$$S = S^{\mathrm{Re}} + iS^{\mathrm{Im}} = \sum_{k=1}^{N} c_{2k}\sin 2k\zeta = y_1 \sin 2\zeta \tag{61}$$

Using the relationships $\sin(ix) = i\sinh x$ and $\cos(ix) = \cosh x$ the trigonometric expansion of the complex functions $\sin 2\zeta$ and $\cos 2\zeta$ are

$$\begin{aligned} \sin 2\zeta &= \sin\left(2\xi + i2\eta\right) & \cos 2\zeta &= \cos\left(2\xi + i2\eta\right) \\ &= \sin 2\xi\cos i2\eta + \cos 2\xi \sin i2\eta & &= \cos 2\xi\cos i2\eta - \sin 2\xi\sin i2\eta \\ &= \sin 2\xi\cosh 2\eta + i\cos 2\xi\sinh 2\eta & &= \cos 2\xi\cosh 2\eta - i\sin 2\xi\sinh 2\eta \end{aligned} \tag{62}$$

Using (62) in (60) and expanding and equating real and imaginary parts gives two recurrence relations

$$\begin{aligned} y_k^{\mathrm{Re}} &= \begin{cases} 0, & \text{for } k > N \\ 2\left(A\, y_{k+1}^{\mathrm{Re}} + B\, y_{k+1}^{\mathrm{Im}}\right) - y_{k+2}^{\mathrm{Re}} + c_{2k}, & \text{for } k = N, N-1, N-2, \ldots, 3, 2, 1 \end{cases} \\ y_k^{\mathrm{Im}} &= \begin{cases} 0, & \text{for } k > N \\ 2\left(A\, y_{k+1}^{\mathrm{Im}} - B\, y_{k+1}^{\mathrm{Re}}\right) - y_{k+2}^{\mathrm{Im}}, & \text{for } k = N, N-1, N-2, \ldots, 3, 2, 1 \end{cases} \end{aligned} \tag{63}$$

where $A = \cos 2\xi\cosh 2\eta$ and $B = \sin 2\xi \sinh 2\eta$ and the complex sum (61) is given as

$$S = S^{\mathrm{Re}} + iS^{\mathrm{Im}} = \sum_{k=1}^{N} c_{2k}\sin 2k\zeta = y_1\sin 2\zeta = \left(y_1^{\mathrm{Re}} + iy_1^{\mathrm{Im}}\right)\left(\sin 2\xi\cosh 2\eta + i\cos 2\xi\sinh 2\eta\right)$$

Expanding and equating real and imaginary parts gives

$$\begin{aligned} S^{\mathrm{Re}} &= y_1^{\mathrm{Re}}\sin 2\xi\cosh 2\eta - y_1^{\mathrm{Im}}\cos 2\xi\sinh 2\eta \\ S^{\mathrm{Im}} &= y_1^{\mathrm{Im}}\sin 2\xi\cosh 2\eta + y_1^{\mathrm{Re}}\cos 2\xi\sinh 2\eta \end{aligned} \tag{64}$$

With suitable changes of variables in equations (63) and (64) the real and imaginary parts of the complex sum S appear in equations (12) and (29) in the *Forward* and *Inverse* transformations.

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